ExACT -- Exact Analytical Conduction Toolbox

R1C11B00T00G11: Temperature and heat flux solutions in two-layer concentric cylinders with prescribed volumetric heat sources[1].

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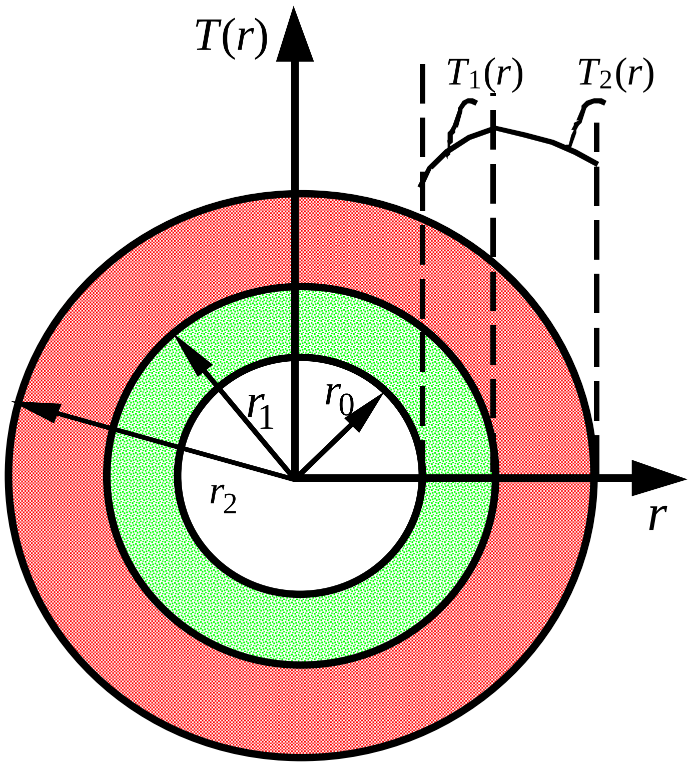

Figure1. Schematic of a two-layer cylindrical body.

1. Governing Equations

Dimensional mathematical statements leading to the temperature solutions for $T_1(r,t)$ and $T_2(r,t)$ in two-layer cylinders, designated as the **R1C11B11T11** problem, are:

$$k_1\left(\frac{\partial^2 T_1}{\partial r^2}+\frac{1}{r}\frac{\partial T_1}{\partial r}\right)+g_1=C_1\frac{\partial T_1}{\partial t}, \quad \text{when } r_0<r<r_1 \qquad \text{(R1C11B00-1a)}$$

in Region 1 and

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$$k_2\left(\frac{\partial^2 T_2}{\partial r^2}+\frac{1}{r}\frac{\partial T_2}{\partial r}\right)+g_2=C_2\frac{\partial T_2}{\partial t}\text{, when } r_1<r<r_2 \quad \text{(R1C11B00-1b)}$$

in Region 2. The parameters C_1 and C_2 are the heat capacitances, k_1 and k_2 are the thermal conductivities, while g_1 and g_2 are the volumetric heat sources. The boundary conditions in dimensional forms are:

$T(r_0,t)=T_{w,i}$ when $r=r_0$, (R1C11B00-2a)

$T_1(r_1,t)=T_2(r_1,t)$ at the contact location, $r=r_1$ (R1C11B00-2b)

$k_1\left[\frac{\partial T_1}{\partial r}(r_1,t)\right]=k_2\left[\frac{\partial T_2}{\partial r}(r_1,t)\right]$ at the contact location, $r=r_1$ (R1C11B00-2c)

$T_2(r_2,t)=T_{w,\text{o}}$ when $r=r_2$ (R1C11B00-2d)

where $T_{w,i}$ and $T_{w,o}$ are the wall temperatures at inside and outside surfaces and, for this case, $T_{w,i}=T_{w,0}=T_w$. The initial conditions are:

$T_1(r,0)=T_i$ (R1C11B00-2e)

$T_2(r,0)=T_i$ (R1C11B00-2f)

2. Dimensional Solutions

Since T_w is a constant, it is appropriate to use reduced temperatures by considering $T_w=0$. Using the separation of variables technique [2], the reduced temperature solutions for $T_1(r,t)$ and $T_2(r,t)$ are:

$$T_1(r,t)=\sum_{j=1}^{\infty} D_j R_{1,j}(\gamma_j r)[1-\exp(-\lambda_j^2 t)]/\lambda_j^2 \quad \text{(R1C11B00-3a)}$$

and

$$T_2(r,t)-T_w=\sum_{j=1}^{\infty} D_j R_{2,j}(r)[1-\exp(-\lambda_j^2 t)]/\lambda_j^2 \quad \text{(R1C11B00-3b)}$$

The eigenfunctions $R_{1,j}(\gamma_j r)$ and $R_{2,j}(\beta_j r)$ have the forms

$R_{1,j}(r)=A_{1,j}J_0(\gamma_j r)+B_{1,j}Y_0(\gamma_j r)$ (R1C11B00-4a)

where

$A_{1,j}=Y_0(\gamma_j r_0)$ and $B_{1,j}=-J_0(\gamma_j r_0)$

while

$R_{2,j}(r)=A_{2,j}J_0(\beta_j r)+B_{2,j}Y_0(\beta_j r)$ (R1C11B00-4b)

with

$$A_{2,j}=\frac{\gamma_j k_1 Y_0(\beta_j r_1)E_{1,j}(\gamma_j)/(\beta_j k_2)-Y_1(\beta_j r_1)E_{0,j}(\gamma_j)}{J_1(\beta_j r_1)Y_0(\beta_j r_1)-J_0(\beta_j r_1)Y_1(\beta_j r_1)} \quad \text{(R1C11B00-4c)}$$

and

$$B_{2,j} = -\frac{\gamma_j k_1 J_0(\beta_j r_1) E_{1,j}(\gamma_j)/(\beta_j k_2) - J_1(\beta_j r_1) E_{0,j}(\gamma_j)}{J_1(\beta_j r_1) Y_0(\beta_j r_1) - J_0(\beta_j r_1) Y_1(\beta_j r_1)} \quad \text{(R1C11B00-4d)}$$

while

$$E_{0,j}(\gamma_j) = A_{1,j} J_0(\gamma_j r_1) + B_{1,j} Y_0(\gamma_j r_1) \quad \text{(R1C11B00-4e)}$$

$$E_{1,j}(\gamma_j) = A_{1,j} J_1(\gamma_j r_1) + B_{1,j} Y_1(\gamma_j r_j) \quad \text{(R1C11B00-4f)}$$

The D_j coefficients are obtainable by using the classical orthogonality conditions and it is presented, as

$$D_j = \frac{P_j}{N_j} \quad \text{(R1C11B00-5a)}$$

The parameter P_j describes the contribution of the volumetric heat sources to these temperature solutions.

$$P_j = \int_{r_0}^{r_1} 2\pi r R_{1,j}(r) g_1(r) dr + \int_{r_1}^{r_2} 2\pi r R_{2,j}(r) g_2(r) dr \quad \text{R1C11B00-5b)}$$

The parameter N_j is the norm defined as

$$N_j = C_1 N_{1,j} + C_2 N_{2,j} \quad \text{(R1C11B00-5c)}$$

with contributions from each of the first and second layers; they are:

$$N_{1,j} = \pi\{[A_{1,j} r_1 J_0(\gamma_j r_1) + B_{1,j} r_1 Y_0(\gamma_j r_1)]^2 + [A_{1,j} r_1 J_1(\gamma_j r_1) + B_{1,j} r_1 Y_1(\gamma_j r_1)]^2 - [A_{1,j} r_0 J_1(\gamma_j r_0) + B_{1,j} r_0 Y_1(\gamma_j r_0)]^2\} \quad \text{(R1C11B00-5d)}$$

$$N_{2,j} = \pi\{[A_{2,j} r_2 J_1(\beta_j r_2) + B_{2,j} r_2 Y_1(\beta_j r_2)]^2 - [A_{2,j} r_1 J_0(\beta_j r_1) + B_{2,j} r_1 Y_0(\beta_j r_1)]^2 - [A_{2,j} r_1 J_1(\beta_j r_1) + B_{2,j} r_1 Y_1(\beta_j r_1)]^2\} \quad \text{(R1C11B00-5e)}$$

The local heat flux values for $q_1(r,t)$ and $q_2(r,t)$ are obtainable using the standard Fourier conduction relation and they become

$$q_1(r,t) = k_1 \sum_{m=1}^{\infty} D_j \gamma_j [A_{1,j} J_1(\gamma_j r) + B_{1,j} Y_1(\gamma_j r)] \exp(-\lambda_j^2 t) \quad \text{(R1C11B00-6a)}$$

$$q_2(r,t) = k_2 \sum_{j=1}^{\infty} D_j \beta_j [A_{2,j} J_1(\beta_j r) + B_{2,j} Y_1(\beta_j r)] \exp(-\lambda_j^2 t) \quad \text{(R1C11B00-6b)}$$

3. Dimensionless Solutions

It is appropriate to modify the aforementioned procedure and reproduce simpler dimensionless forms of these equations. The reduced initial conditions are $\tilde{T}_{i,1} = \tilde{T}_{i,2} = 0$, in this formulation. This suggests selecting the dimensionless temperatures $\tilde{T}_1(r,t) = [T_1(r,t) - T_w]/T_r$ and $\tilde{T}_2(r,t) = [T_2(r,t) - T_w]/T_r$ where the surface temperature

$T_w = T_{w,i} = T_{w,o}$ and T_r is a reference temperature. The dimensionless parameters for the **R1C11B00T00G11** problem are selected as

$$\tilde{T}_1(\tilde{r},\tilde{t}) = \frac{T_1(r,t) - T_w}{T_r},\ \tilde{T}_2(\tilde{r},\tilde{t}) = \frac{T_2(r,t) - T_w}{T_r},\ \tilde{q}_1(\tilde{r},\tilde{t}) = \frac{r_2 q_1(r,t)}{k_2 T_r},$$

$$\tilde{q}_2(\tilde{r},\tilde{t}) = \frac{r_2 q_2(r,t)}{k_2 T_r},\ \tilde{g}_1 = \frac{r_2^2 g_1}{k_2 T_r},\ \tilde{g}_2 = \frac{r_2^2 g_2}{k_2 T_r},\ \tilde{r} = r/r_2,$$

$$\tilde{t} = \alpha_2 t / r_2^2,\ \tilde{C} = C_1/C_2,\ \tilde{k} = k_1/k_2,\ \tilde{\alpha} = \alpha_1/\alpha_2,\ \tilde{r}_1 = r_1/r_2 \qquad \text{(R1C11B00-7)}$$

Then, the dimensionless mathematical relation forms for this **R1C11B00T00G11** problem are:

$$\frac{\partial^2 \tilde{T}_1}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \tilde{T}_1}{\partial \tilde{r}} + \frac{\tilde{g}_1}{\tilde{k}} = \frac{1}{\tilde{\alpha}}\frac{\partial \tilde{T}_1}{\partial \tilde{t}},\ \text{when } r_0 < \tilde{r} < \tilde{r}_1 \qquad \text{(R1C11B00-8a)}$$

in Region 1 and

$$\frac{\partial^2 \tilde{T}_2}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \tilde{T}_2}{\partial \tilde{r}} + \tilde{g}_2 = \frac{1}{\tilde{\alpha}}\frac{\partial \tilde{T}_2}{\partial \tilde{t}},\ \text{when } \tilde{r}_1 < \tilde{r} < 1 \qquad \text{(R1C11B00-8b)}$$

in Region 2.

The boundary conditions are:

$$\tilde{T}_1(r_0,\tilde{t}) = 0 \qquad \text{(R1C11B00-9a)}$$

$$\tilde{T}_1(\tilde{r}_1,\tilde{t}) = \tilde{T}_2(\tilde{r}_1,\tilde{t}) \qquad \text{(R1C11B00-9b)}$$

$$\left[\frac{\partial \tilde{T}_1}{\partial \tilde{r}}(\tilde{r}_1,\tilde{t})\right] = \tilde{k}_2\left[\frac{\partial \tilde{T}_2}{\partial \tilde{r}_1}(\tilde{r}_1,\tilde{t})\right] \qquad \text{(R1C11B00-9c)}$$

$$\tilde{T}_2(1,\tilde{t}) = 0 \qquad \text{(R1C11B00-9d)}$$

while the initial conditions become

$$\tilde{T}_1(\tilde{r},0) = \tilde{T}_{i,1} = 0 \qquad \text{(R1C11B00-9e)}$$

$$\tilde{T}_2(\tilde{r},0) = \tilde{T}_{i,2} = 0 \qquad \text{(R1C11B00-9f)}$$

The parameter T_r can be selected as $T_r = g_2 r_2^2/k_2$ if $g_2 \neq 0$ or equal to $g_1 r_2^2/k_2$. Furthermore, to use the same superposition for two solutions, T_r can be selected as a reference temperature employed in **R1C11B00T11G00** case.

For better accuracy, the solution is determined using the steady state and complementary transient solution. The acquired set of dimensionless solutions is valid for all values of $\tilde{t} > 0$, that is

$$\tilde{T}_1(\tilde{r},\tilde{t}) = \tilde{T}_{gs,1} - \sum_{j=1}^{\infty} \frac{P_j R_{1,j}(\tilde{r})\exp(-\lambda_j^2 \tilde{t})}{N_j \lambda_j^2} \qquad \text{(R1C11B00-10a)}$$

$$\tilde{T}_2(\tilde{r},\tilde{t}) = \tilde{T}_{gs,2} - \sum_{j=1}^{\infty} \frac{P_j R_{2,j}(\tilde{r})\exp(-\lambda_j^2 \tilde{t})}{N_j \lambda_j^2} \qquad \text{(R1C11B00-10b)}$$

The functions $R_{1,j}(\tilde{r})$ and $R_{2,j}(\tilde{r})$ have the forms

$$R_{1,j} = A_{1,j} J_0(\gamma_j \tilde{r}) + B_{1,j} Y_0(\gamma_j \tilde{r}) \text{ when } \tilde{r} \le \tilde{r}_1 \qquad \text{(R1C11B00-11a)}$$

and

$$R_{2,j}(\tilde{r}) = A_{2j} J_0(\beta_j \tilde{r}) + B_{2j} Y_0(\beta_j \tilde{r}) \text{ when } \tilde{r}_1 < \tilde{r} \le \tilde{r}_2 = 1 \qquad \text{(R1C11B00-11b)}$$

Wherein, in dimensionless form, the coefficients $A_{1,j}$, $B_{1,j}$, $A_{2,j}$, and $B_{2,j}$ are

$$A_{1,j} = Y_0(\gamma_j \tilde{r}_0),$$

$$B_{1,j} = -J_0(\gamma_j \tilde{r}_0),$$

$$A_{2,j} = \frac{\gamma_j k_1 Y_0(\beta_j \tilde{r}_1) J_1(\gamma_j \tilde{r}_1)/(\beta_j k_2) - Y_1(\beta_j \tilde{r}_1) J_0(\gamma_j \tilde{r}_1)}{J_1(\beta_j \tilde{r}_1) Y_0(\beta_j \tilde{r}_1) - J_0(\beta_j \tilde{r}_1) Y_1(\beta_j \tilde{r}_1)},$$

and

$$B_{2,j} = -\frac{\gamma_j k_1 J_0(\beta_j \tilde{r}_1) J_1(\gamma_j \tilde{r}_1)/(\beta_j k_2) - J_1(\beta_j \tilde{r}_1) J_0(\gamma_j \tilde{r}_1)}{J_1(\beta_j \tilde{r}_1) Y_0(\beta_j \tilde{r}_1) - J_0(\beta_j \tilde{r}_1) Y_1(\beta_j \tilde{r}_1)}$$

The members of other dimensionless parameter are:

$$N_j = \tilde{C}_1 N_{1,j} + N_{2.j} \qquad \text{(R1C11B11-11c)}$$

with

$$N_{1,j} = \pi\{[A_{1,j}\tilde{r}_1 J_0(\gamma_j \tilde{r}_0) + B_{1,j}\tilde{r}_1 Y_0(\gamma_j \tilde{r}_0)]^2 + [A_{1,j}\tilde{r}_1 J_1(\gamma_j \tilde{r}_1) + B_{1,j}\tilde{r}_1 Y_1(\gamma_j \tilde{r}_1)]^2$$
$$- [A_{1,j}\tilde{r}_0 J_1(\gamma_j \tilde{r}_0) + B_{1,j}\tilde{r}_0 Y_1(\gamma_j \tilde{r}_0)]^2\} \qquad \text{(R1C11B11-11d)}$$

$$N_{2,j} = \pi\{[A_{2,j} J_1(\beta_j) + B_{2,j} Y_1(\beta_j)]^2 - [A_{2,j}\tilde{r}_1 J_0(\beta_j \tilde{r}_1) + B_{2,j}\tilde{r}_1 Y_0(\beta_j \tilde{r}_1)]^2\}$$
$$- [A_{2,j}\tilde{r}_1 J_1(\beta_j \tilde{r}_1) + B_{2,j}\tilde{r}_1 Y_1(\beta_j \tilde{r}_1)]^2 \qquad \text{(R1C11B11-11e)}$$

$$P_j = \left(\frac{2\pi\tilde{g}_1}{\gamma_j}\right)\Big[A_{1,j}\tilde{r}_1 J_1(\gamma_j \tilde{r}_1) + B_{1,j}\tilde{r}_1 Y_1(\gamma_j \tilde{r}_1)$$
$$- A_{1,j}\tilde{r}_0 J_1(\gamma_j \tilde{r}_0) - B_{1,j}\tilde{r}_0 Y_1(\gamma_j \tilde{r}_0)$$
$$+ \left(\frac{2\pi\tilde{g}_2}{\beta_j}\right)\Big[A_{2,j} J_1(\beta_j) + B_{2,j} Y_1(\beta_j) \qquad \text{(R1C11B00-11f)}$$
$$- A_{2,j}\tilde{r}_1 J_1(\beta_j \tilde{r}_1) - B_{2,j}\tilde{r}_1 Y_1(\beta_j \tilde{r}_1)$$

with $\tilde{r}_2 = 1$. The steady state temperatures $\tilde{T}_{gs,1}$ and $\tilde{T}_{gs,2}$ are

$$\tilde{T}_{sg,1} = -\frac{\tilde{g}_1(\tilde{r}^2 - \tilde{r}_0^2)}{4\tilde{k}} + \frac{1}{4\tilde{k}\ln(\tilde{r}_1) - 4\ln(\tilde{r}_1/\tilde{r}_0)}\{\ln(\tilde{r}/\tilde{r}_0) \qquad \text{(R1C11B00-11g)}$$
$$\times[\tilde{g}_1(\tilde{r}_0^2 - \tilde{r}_1^2) - \tilde{g}_2(1 - \tilde{r}_1^2) + 2(\tilde{g}_1 - \tilde{g}_2)\tilde{r}_1^2 \ln(\tilde{r}_1/\tilde{r}_0)]\}$$

$$\tilde{T}_{sg,2} = \frac{\tilde{g}_2(1-\tilde{r}^2)}{4} + \frac{1}{4\tilde{k}\ln(\tilde{r}_1) - 4\ln(\tilde{r}_1/\tilde{r}_0)}\{\ln(\tilde{r}) \times[\tilde{g}_1(\tilde{r}_0^2 - \tilde{r}_1^2) - \tilde{k}\tilde{g}_2(1-\tilde{r}_1^2) + 2(\tilde{g}_1 - \tilde{g}_2)\tilde{r}_1^2 \ln(\tilde{r}_1/\tilde{r}_0)]\} \quad \text{(R1C11B00-11h)}$$

4. Determination of Eigenvalues

The relation between parameters γ_j and λ_j, for a given index *j*, is obtainable by placing $T_1(\tilde{r},\tilde{t})$ from Eq. (R1C11B00-10a) into energy equation, Eq. (R1C11B00-8a). The process is repeated by placing $T_2(\tilde{r},\tilde{t})$ from Eq. (R1C11B00-10b) into Eq. (R1C11B00-8b) to get a relation between parameters β_j and λ_j; the results are

$$\gamma_j = \lambda_j / \sqrt{\tilde{\alpha}_1} \quad \text{(R1C11B00-12a)}$$

$$\beta_j = \lambda_j \quad \text{(R1C11B00-12b)}$$

The eigenvalues are found using the eigencondition obtained from the boundary condition at $\tilde{r} = 1$ and after replacing parameters γ_j and β_j with the parameter λ_j in equation (R1C11B00-11b) to get

$$f = A_{2,j}J_0(\lambda_j) + B_{2,j}Y_0(\lambda_j) \Rightarrow 0 \quad \text{(R1C11B00-13)}$$

This eigencondition is be satisfied at designated deterministic λ_j locations called eigenvalues. The number of needed eigenvalues depends on the value of $\tilde{t}$ and the desired accuracy of the acquired temperature solution. As an example for 10 accurate digits when $\tilde{t} = 0.01$, the maximum value of λ_j for a specified error of ε is obtainable from the relation

$$\exp[-\lambda_j^2\tilde{t}] \approx \varepsilon \quad \text{(R1C11B00-14a)}$$

As an illustration, by setting $\lambda_j = \lambda_{max}$ while $\lambda_{max}^2\tilde{t} = K_{max}$, when $\varepsilon = 10^{-10}$ and $\tilde{t} = 0.01$, it produces $K_{max} = 23$ and $\lambda_{max} \approx 48$. By defining $m_{max} = m_1 + m_2$ as the number of needed eigenvalues, the contributions of layers 1 and 2 are

$$m_1 = [\lambda_{max}(r_1/r_2 - r_0/r_2)/\sqrt{\tilde{\alpha}_1}]/\pi = [\lambda_{max}\tilde{r}_1)/\sqrt{\tilde{\alpha}_1}]/\pi \quad \text{(R1C11B00-14b)}$$

$$m_2 = [\lambda_{max}(1 - r_1/r_2)/\sqrt{\tilde{\alpha}_2}]/\pi = \lambda_{max}(1-\tilde{r}_1)/\pi \quad \text{(R1C11B00-14c)}$$

and the total number of needed eigenvalues to satisfy this condition becomes

$$m_{max} \cong \frac{1}{\pi}\sqrt{\frac{K_{max}}{\tilde{t}}}\left[1 - \tilde{r}_1 + \tilde{r}_1/\sqrt{\tilde{\alpha}_1}\right] \quad \text{(R1C11B00-14d)}$$

The accompanying program will produce the needed number of terms, closely related to m_{max}, for a given value of $\lambda_{max} \cong \sqrt{K_{max}/\tilde{t}}$; the default value in the accompanying program is $\lambda_{max} \approx 48$.

5. Dimensionless Heat Flux

Once the temperature is known, the heat flux is obtainable through the Fourier equation,

$$\tilde{q}_1(\tilde{r},\tilde{t}) = \tilde{q}_{gs,1} - \sum_{j=1}^{\infty}\left(\tilde{k}\frac{dR_{1,j}(\tilde{r})}{d\tilde{r}}\right)\frac{P_j\exp(-\lambda_j^2\tilde{t})}{N_j} \qquad \text{(R1C11B00-15a)}$$

$$\tilde{q}_2(\tilde{r},\tilde{t}) = \tilde{q}_{gs,2} - \sum_{j=1}^{\infty}\left(\frac{dR_{2,j}(\tilde{r})}{d\tilde{r}}\right)\frac{\exp(-\lambda_j^2\tilde{t})}{N_j} \qquad \text{(R1C11B00-15b)}$$

where

$$dR_{1,j}(\tilde{r})/d\tilde{x} = -A_1\gamma_j J_1(\gamma_j\tilde{r}) - B_1\gamma_j Y_1(\gamma_j\tilde{r}) \qquad \text{(R1C11B00-16a)}$$

$$dR_{2,j}(\tilde{r})/d\tilde{r} = -A_2\beta_j J_1(\beta_j\tilde{r}) - B_2\beta_j Y_1(\beta_j\tilde{r}) \qquad \text{(R1C11B00-16b)}$$

$$\tilde{q}_{sg,1} = \frac{\tilde{g}_1\tilde{r}}{2} - \frac{\tilde{k}}{4\tilde{k}\ln(\tilde{r}_1) - 4\ln(\tilde{r}_1/\tilde{r}_0)} \times\{[\tilde{g}_1(\tilde{r}_0^2 - \tilde{r}_1^2) - \tilde{g}_2(1-\tilde{r}_1^2) + 2(\tilde{g}_1 - \tilde{g}_2)\tilde{r}_1^2\ln(\tilde{r}_1/\tilde{r}_0)]/\tilde{r}\} \qquad \text{(R1C11B00-16c)}$$

$$\tilde{T}_{sg,2} = \frac{\tilde{g}_2\tilde{r}}{2} - \frac{1}{4\tilde{k}\ln(\tilde{r}_1) - 4\ln(\tilde{r}_1/\tilde{r}_0)} \times\{[\tilde{g}_1(\tilde{r}_0^2 - \tilde{r}_1^2) - \tilde{k}\tilde{g}_2(1-\tilde{r}_1^2) + 2(\tilde{g}_1 - \tilde{g}_2)\tilde{r}_1^2\ln(\tilde{r}_1/\tilde{r}_0)]/\tilde{r}\} \qquad \text{(R1C11B00-16d)}$$

6. Discussion

To show the behavior of this solution, temperature and heat flux values are numerically obtained for the **R1C11B00T00G11** case when $\tilde{g}_1 = 2$ and $\tilde{g}_2 = 1$. Samples of acquired temperature values are in Table 1(a) and heat flux values are in Table 1(b). The presented temperature and heat flux values in Tables 1(a) and 1(b) are computed having $k_1/k_2 = 2$, $C_1/C_2 = 1$, and $\tilde{r}_1 = r_1/r_2 = 1/2$.

It is of interest to compare the results from this solution to those from a similar temperature solution for a solid cylinder. Accordingly, when $\tilde{k}_1 = k_1/k_2 = 2$, $\tilde{C}_1 = 1$, $\tilde{g}_1 = 2$, $\tilde{g}_2 = 1$, $\tilde{r}_0 = 2/10$ and $\tilde{r}_1 = 1/2$, the acquired two-layer temperature and heat flux data are at $\tilde{r}_1 = 0.8$, 0.85, 0.9, 0.95 and 1.0, presented in Tables 2(a) and 2(b). Also, data are obtained using the classical series solution for a single-layer solid cylinder, with $k = k_2$ and $C = C_2$, located below two-layer solutions. Both sets agreed well with when $\tilde{t} < 0.01$. Numerical differences are detected at larger values of $\tilde{t}$. The computed values of heat flux for a single-layer cylinder show similar agreement but with slightly larger differences, as shown in Table 2(b).

Furthermore, for verification of the solutions, estimated small time temperature and heat flux values are prepared with a Laplace transform technique using the methodology presented in [4], see Appendix. There is a good agreement between series solutions and the small time solutions when $\tilde{t} < 0.01$. When $\tilde{k} = 2$, $\tilde{C} = 1$, $\tilde{g}_1 = \tilde{g}_2 = 1$, $\tilde{r}_0 = r_0 / r_2 = 1/5$, $\tilde{r}_1 = r_1 / r_2 = 1/2$, $\tilde{t} = 0.01$ and $\tilde{r} = 0.95$, the attached two-layer program would produces $\tilde{T}_2 = 0.00436335$ and $\tilde{q}_2 = 0.06864781$ while a small time Laplace transfer technique, when $k = k_2$, produced $\tilde{T}_2 = 0.00436337$ and $\tilde{q}_2 = 0.06864793$, with expected minor differences. The deviations are expected to increase at larger time although they become slightly larger at $\tilde{r} = 0.95$ when $\tilde{t} = 0.02$. At this location and this time, the series solution yields $\tilde{T}_2 = 0.00645813$ and $\tilde{q}_2 = 0.11024862$ while the approximate small time solution produced $\tilde{T}_2 = 0.00645946$ and $\tilde{q}_2 = 0.11028172$. The functional form of the small time solution for temperature is in the Appendix.

The graphical presentations for temperature and heat flux values are in Figures 2 and 3. The plotted data are for $\tilde{k}_1 = 2$, $\tilde{C}_1 = 1$, $\tilde{r}_0 = r_0 / r_2 = 1/5$, $\tilde{r}_1 = r_1 / r_2 = 1/2$, $\tilde{g}_1 = 2$ and $\tilde{g}_2 = 1$. These graphs demonstrate expected behaviors of temperature and heat flux solutions.

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Appendix

A small time solution is prepared for determination of temperature for **R0B0T0G1** case when time is small. The technique presented in Carslaw and Jaeger [4, Section 13.3, Eq. (1-3)] is used for this study. Equation (3) in [4] is written in dimensionless space as

$$\psi(\tilde{r},\eta)=\left(\frac{1}{\tilde{r}}\right)\text{erfc}(\eta)+\left(\frac{(1-\tilde{r})\sqrt{\tilde{t}}}{4\sqrt{\tilde{r}^3}}\right)\text{ierf}(\eta) + \left(\frac{(9-2\tilde{r}-7\tilde{r}^2)\tilde{t}}{32\sqrt{\tilde{r}^5}}\right)\text{i}^2\text{erf}(\eta) \qquad \text{(R1C11B00-A1)}$$

where $\eta=(1-\tilde{r})/(2\sqrt{\tilde{t}})$ and the functions $\text{ierf}(\eta)$ and $\text{i}^2\text{erf}(\eta)$ are presented in [1, Eqs. (E-14a) and (E-14b)]. Then, the temperature solution for a uniform volumetric heat source is obtainable by using the convolution theorem as

$$\tilde{T}=\tilde{g}\,\tilde{t}-\int_0^{\tilde{t}}\tilde{g}\psi(\tilde{r},\tau)d\tau \qquad \text{(R1C11B00-A2)}$$

Exact integration of this function is possible. After performing the integration and some algebraic manipulations, the result is prepared for this study as

$$\begin{aligned}\tilde{T}=\tilde{g}\,\tilde{t}-\frac{\tilde{g}\exp[-1/(4\text{Fo})]}{3072\sqrt{\pi\tilde{r}^5}}\Big[&-\sqrt{4\text{Fo}}\ \{(1-\tilde{r})^2[9(5+2\tilde{t})-4\tilde{r}(73+\tilde{t})+2\tilde{r}^2(975-7\tilde{t})\\ &-132\tilde{r}^3-35\tilde{r}^4]\}+8\sqrt{\tilde{t}}(9-43\tilde{r}+27\tilde{r}^2+7\tilde{r}^3)(1-2\tilde{r}+\tilde{r}^2-2\tilde{t})\Big]\\ +\frac{\tilde{g}\text{erfc}\,[1/\sqrt{4\text{Fo}}]}{3072\sqrt{\tilde{r}^5}}\Big[&4(1-\tilde{r})^4(9-34\tilde{r}-7\tilde{r}^2)-9(5+12\tilde{t}+12\tilde{t}^2)\\ &+\tilde{r}(382+624\tilde{t}+24\tilde{t}^2)+\tilde{r}^2(-2579-3912\tilde{t}+84\tilde{t}^2)\\ &+4\tilde{r}^3(1081+60\tilde{t})+\tilde{r}^4(-2179+84\tilde{t})+62\tilde{r}^5+35\tilde{r}^6\Big]\end{aligned} \qquad \text{(R1C11B00-A3)}$$

For convenience of a user, below is the small time temperature solution "Tsm" in Mathematica form where "t" stands for $\tilde{t}$, "r" stands for $\tilde{r}$, and $\text{Fo}=\tilde{t}/(1-\tilde{r})^2$. It provides small time temperature solution for a selected set of volumetric heat source "g", radial location "r", and time "t", all in dimensionless form.

```
Tsm=g*t-g*Exp[-1/4/Fo]*(-Sqrt[4*Fo]*((1-r)^2*(9*(5+2*t)-
4*r*(73+t)+2*r^2*(975-7*t)-132*r^3-35*r^4))+8*Sqrt[t]*(9-
43*r+27*r^2+7*r^3)*(1-2*r+r^2-
2*t))/3072/Sqrt[Pi*r^5]+g*Erfc[1/Sqrt[4*Fo]]*(4*(1-r)^4*(9-
34*r-7*r^2)-9*(5+12*t+12*t^2)+r*(382+624*t+24*t^2)+r^2*(-
2579-3912*t+84*t^2)+4*r^3*(1081+60*t)+r^4*(-
2179+84*t)+62*r^5+35*r^6)/3072/Sqrt[r^5];
```

Table 1(a). Variation of temperature at different locations with time having physical properties $\tilde{k} = 2$, $\tilde{C} = 1$, $\tilde{r}_0 = 1/5$, $\tilde{r}_1 = 1/2$, while $\tilde{g}_1 = 2$, and $\tilde{g}_2 = 1$.

$\alpha t / r_2^2$	r / r_2 =0.20	r / r_2 =0.35	r / r_2 =0.5	r / r_2 =0.75	r / r_2 =1.00
0.0000	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.0001	0.0000000000	0.0002000000	0.0001581361	0.0001000000	0.0000000000
0.0010	0.0000000000	0.0019925955	0.0015716539	0.0010000000	0.0000000000
0.0050	0.0000000000	0.0086458433	0.0077065665	0.0049917177	0.0000000000
0.010	0.0000000000	0.0146641122	0.0146882979	0.0098414200	0.0000000000
0.025	0.0000000000	0.0268066255	0.0310649116	0.0226111511	0.0000000000
0.050	0.0000000000	0.0391724557	0.0488726351	0.0379389295	0.0000000000
0.075	0.0000000000	0.0466178206	0.0597275918	0.0476455438	0.0000000000
0.100	0.0000000000	0.0512139326	0.0664448269	0.0537121362	0.0000000000
0.125	0.0000000000	0.0540673378	0.0706177499	0.0574909213	0.0000000000
0.150	0.0000000000	0.0558414958	0.0732127875	0.0598425394	0.0000000000
0.175	0.0000000000	0.0569450600	0.0748270308	0.0613056456	0.0000000000
0.200	0.0000000000	0.0576315756	0.0758312468	0.0622158870	0.0000000000
0.225	0.0000000000	0.0580586622	0.0764559794	0.0627821649	0.0000000000
0.250	0.0000000000	0.0583243582	0.0768446338	0.0631344552	0.0000000000
0.275	0.0000000000	0.0584896513	0.0770864211	0.0633536201	0.0000000000
0.300	0.0000000000	0.0585924825	0.0772368405	0.0634899658	0.0000000000
0.325	0.0000000000	0.0586564553	0.0773304186	0.0635747884	0.0000000000
0.350	0.0000000000	0.0586962537	0.0773886348	0.0636275578	0.0000000000
0.375	0.0000000000	0.0587210129	0.0774248520	0.0636603864	0.0000000000
0.400	0.0000000000	0.0587364159	0.0774473832	0.0636808095	0.0000000000
0.500	0.0000000000	0.0587579756	0.0774789203	0.0637093959	0.0000000000
∞	0.0000000000	0.0587617740	0.0774844765	0.0637144322	0.0000000000

Table 1(b). Variation of heat flux at different locations with time having physical properties $\tilde{k} = 2$, $\tilde{C} = 1$, $\tilde{r}_0 = 1/5$, $\tilde{r}_1 = 1/2$, while $\tilde{g}_1 = 2$, and $\tilde{g}_2 = 1$.

$\alpha t / r_2^2$	r/r_2 =0.20	r/r_2 =0.35	r/r_2 =0.5	r/r_2 =0.75	r/r_2 =1.00
0.0000	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.0001	-0.0329026709	0.0000000000	0.0066089520	0.0000000000	0.0112336970
0.0010	-0.1105567282	-0.0004006333	0.0208721336	0.0000000002	0.0351794444
0.0050	-0.2696603138	-0.0247760599	0.0444708734	0.0006440029	0.0772535439
0.010	-0.3926752924	-0.0654571715	0.0516074320	0.0068958440	0.1077484688
0.025	-0.6250918577	-0.1675265687	0.0340819023	0.0414300560	0.1668235603
0.050	-0.8555062216	-0.2799098397	-0.0023425620	0.0879396731	0.2314376385
0.075	-0.9935938967	-0.3484701358	-0.0267054356	0.1165900435	0.2726635932
0.100	-1.078763795	-0.3908971970	-0.0420594085	0.1342945779	0.2985450313
0.125	-1.131628028	-0.4172540083	-0.0516431315	0.1452878357	0.3146863250
0.150	-1.164495380	-0.4336446563	-0.0576106063	0.1521233730	0.3247347756
0.175	-1.184939245	-0.4438404597	-0.0613239453	0.1563752689	0.3309871860
0.200	-1.197657101	-0.4501832393	-0.0636342155	0.1590203358	0.3348770888
0.225	-1.205568967	-0.4541291444	-0.0650714927	0.1606658535	0.3372970849
0.250	-1.210491038	-0.4565839439	-0.0659656477	0.1616895510	0.3388026045
0.275	-1.213553126	-0.4581111091	-0.0665219151	0.1623264075	0.3397392109
0.300	-1.215458095	-0.4590611805	-0.0668679772	0.1627226050	0.3403218874
0.325	-1.216643203	-0.4596522336	-0.0670832674	0.1629690852	0.3406843788
0.350	-1.217380476	-0.4600199362	-0.0672172025	0.1631224240	0.3409098899
0.375	-1.217839144	-0.4602486893	-0.0673005255	0.1632178184	0.3410501837
0.400	-1.218124488	-0.4603909999	-0.0673523620	0.1632771645	0.3411374624
0.500	-1.218523886	-0.4605901927	-0.0674249177	0.1633602317	0.3412596269
∞	-1.218594251	-0.4606252864	-0.06743770046	0.1633748664	0.3412811498

Table 2(a). Temperature solutions for a two layer system when $\tilde{k}=2$, $\tilde{C}=1$, $\tilde{r}_0=1/5$, $\tilde{r}_1=1/2$, $\tilde{g}_1=2$, and $\tilde{g}_2=1$ with their comparison to a single-layer solution values when $\tilde{k}=1$, $\tilde{C}=1$, $\tilde{r}_0=0$, and $\tilde{g}=1$.

$\alpha t/r_2^2$	r/r_2 =0.8	r/r_2 =0.85	r/r_2 =0.9	r/r_2 =0.95	r/r_2 =1.0
0.0001	0.0001000000 0.0001000000	0.0001000000 0.0001000000	0.0001000000 0.0001000000	0.0000999951 0.0000999951	0.00 0.00
0.0002	0.0002000000 0.0002000000	0.0002000000 0.0002000000	0.0002000000 0.0002000000	0.0001995078 0.0001995078	0.00 0.00
0.0005	0.0005000000 0.0005000000	0.0004999999 0.0004999999	0.0004998847 0.0004998847	0.0004810092 0.0004810092	0.00 0.00
0.0010	0.0009999993 0.0009999993	0.0009998894 0.0009998894	0.0009940600 0.0009940600	0.0008814788 0.0008814788	0.00 0.00
0.0020	0.0019995109 0.0019995108	0.0019920608 0.0019920608	0.0019219364 0.0019219364	0.0015157493 0.0015157493	0.00 0.00
0.0050	0.0049363580 0.0049354380	0.0047521107 0.0047519878	0.0042053982 0.0042053849	0.0028484636 0.0028484625	0.00 0.00
0.010	0.0093992356 0.0093639889	0.0085819602 0.0085707620	0.0070505379 0.0070473321	0.0043641430 0.0043633542	0.00 0.00
0.020	0.0169668889 0.0166213782	0.0146009749 0.0144239934	0.0112323045 0.0111483273	0.0064920229 0.0064593380	0.00 0.00
0.050	0.0330213583 0.0317497828	0.0269555553 0.0260462492	0.0195615840 0.0189836095	0.0106433449 0.0103660526	0.00 0.00
0.10	0.0462083981 0.0478063967	0.0371184756 0.0380805443	0.0264155987 0.0269477733	0.0140589914 0.0142917149	0.00 0.00

Table 2(b). Heat flux solutions for a two layer system when $\tilde{k}=2$, $\tilde{C}=1$, $\tilde{r}_0=1/5$, $\tilde{r}_1=1/2$, $\tilde{g}_1=2$, and $\tilde{g}_2=1$ with their comparison to a single-layer solution values when $\tilde{k}=1$, $\tilde{C}=1$, $\tilde{r}_0=0$, and $\tilde{g}=1$.

$\alpha t/r_2^2$	r/r_2=0.8	r/r_2=0.85	r/r_2=0.9	r/r_2=0.95	r/r_2=1.0
0.0001	0.0000000000 0.0000000000	0.0000000000 0.0000000000	0.0000000000 0.0000000000	0.0000014700 0.0000014700	0.0112336970 0.0112336970
0.0002	0.0000000000 0.0000000000	0.0000000000 0.0000000000	0.0000000022 0.0000000022	0.0000819919 0.0000819919	0.0158574227 0.0158574227
0.0005	0.0000000000 0.0000000000	0.0000000140 0.0000000140	0.0000141330 0.0000141330	0.0015666006 0.0015666006	0.0249802580 0.0249802580
0.0010	0.0000000791 0.0000000791	0.0000099774 0.0000099774	0.0004123627 0.0004123627	0.0060137941 0.0060137941	0.0351794444 0.0351794444
0.0020	0.0000298336 0.0000298216	0.0004018024 0.0004018022	0.0031968668 0.0031968668	0.0156076594 0.0156076594	0.0494539789 0.0494539789
0.0050	0.0018599074 0.0018599074	0.0062157530 0.0062157530	0.0171293563 0.0171293563	0.0394550610 0.0394550610	0.0772535351 0.0772535351
0.010	0.0116269824 0.0108540735	0.0221875558 0.0219193824	0.0405454135 0.0404615636	0.0686725522 0.0686478591	0.1077484688 0.1077369691
0.020	0.0396270057 0.0352056855	0.0561527005 0.0536722723	0.0798288721 0.0784777732	0.1110564754 0.1102804482	0.1498685242 0.1492819544
0.050	0.1092728243 0.1017719157	0.1339631425 0.1270114444	0.1624531885 0.1561488601	0.1949493665 0.1892007667	0.2314376385 0.2260604990
0.10	0.1658772875 0.1812356337	0.1978111155 0.2081936749	0.2304419215 0.2375057318	0.2639863785 0.2691118217	0.2985450313 0.3029120970

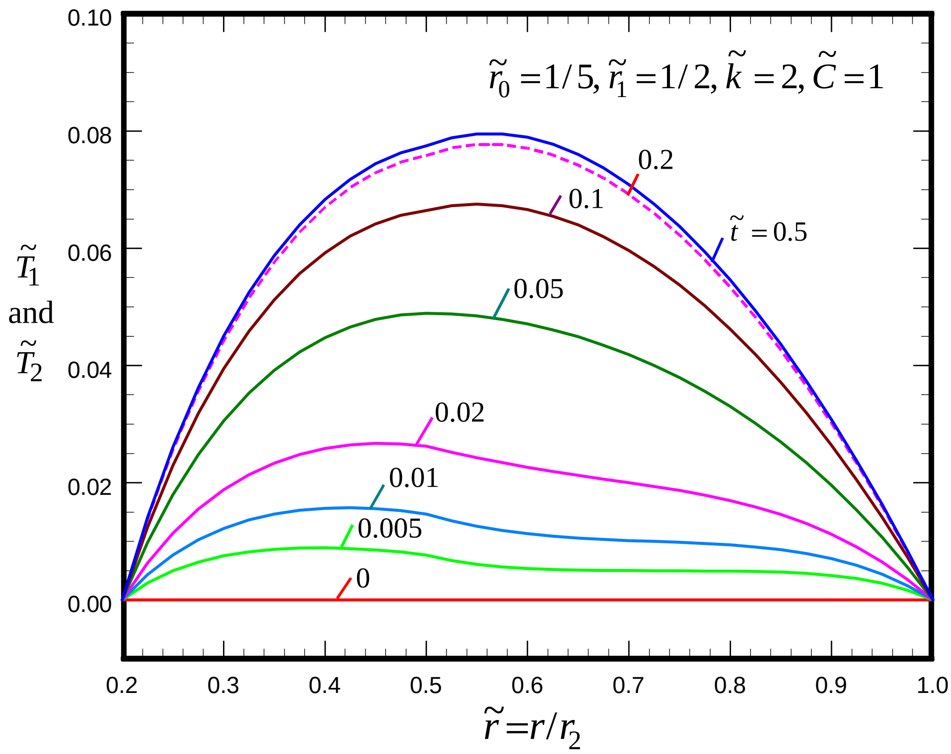

Figure 2. Temperature variations at different selected times as a function of $\tilde{r} = r/r_2$ when $\tilde{g}_1 = 10$ and. $\tilde{g}_2 = 1$.

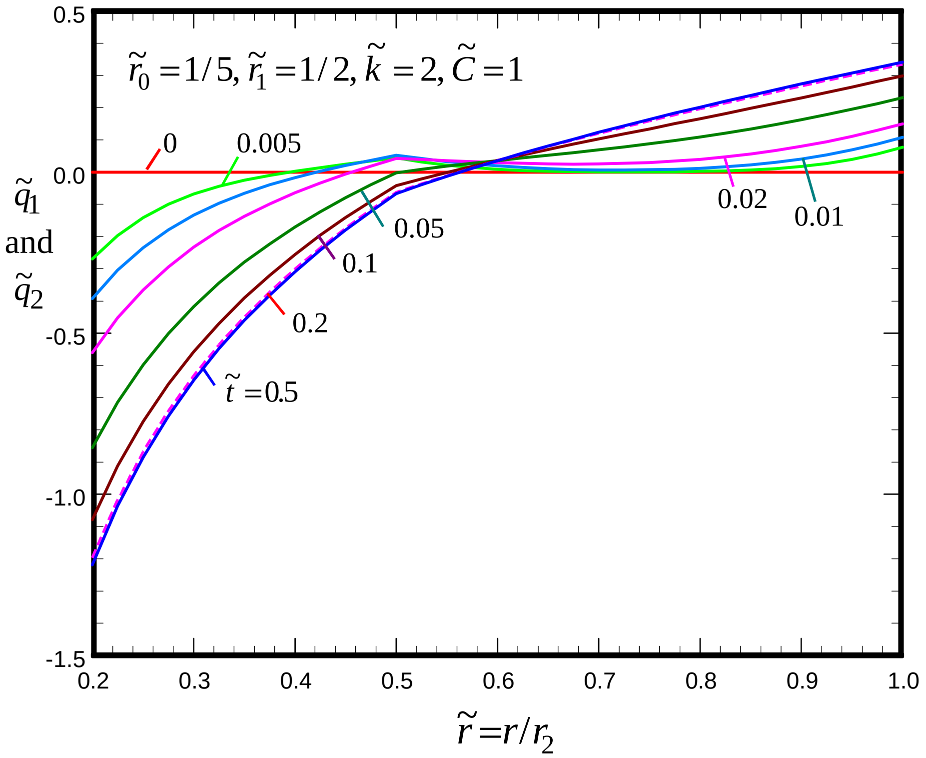

Figure 3. Heat flux variations at different selected times as a function of $\tilde{r} = r/r_2$ when $\tilde{g}_1 = 10$ and. $\tilde{g}_2 = 1$.

Below is a Mathematical program for determination of temperature, for **R1C11B00T00G11** case, by using the separation of variables. The input data are to be placed within the third line of the program below.

The input data to be selected are:
r1: The inner radius.
r2: The outer radius.
k1 and k2: Thermal conductivities in Regions 1 and 2.
Cp1 and Cp2: Thermal capacitances in Regions 1 and 2.
g1 and g2: volumetric heat sources in Regions 1 and 2.
Lambmax: Default is 68 for time tm=0.005; otherwise, get of λ_{max} from Eq. (14a)

(*R1C13B00T00G11 Program*)

```
Clear[lamb,gama,beta];
r2=1;r1=1/2;r0=2/10;k1=2;k2=1;Cp1=1;Cp2=1;g1=2;g2=1;
lambmax=68;
Tw1=0;Tw2=0;Ti1=0;Ti2=0;
Avd=(1-(r1/r2)*(1-1/Sqrt[k1/Cp1]))/Pi;df0=1/4/Avd;
m1=Round[0.5+lambmax*(r1/r2-r0/r2)/Pi/Sqrt[k1/Cp1]];
m2=Round[0.5+lambmax*(1-r1/r2)/Pi/Sqrt[k2/Cp2]];
Nterm=m1+m2;
e1=N[(2404*Sqrt[k1/Cp1]/1000/r1),30];
e2=N[(2404*Sqrt[k2/Cp2]/1000/r2),30];
df=If[e1<e2,e1/2,e2/2];
gama=(lamb/Sqrt[k1/Cp1]);beta=(lamb/Sqrt[k2/Cp2]);
A1=BesselY[0,gama*r0];B1=-BesselJ[0,gama*r0];
R10=A1 BesselJ[0,gama r1]+B1 BesselY[0,gama r1];
R11=A1 gama k1 BesselJ[1,gama r1]+B1 gama k1 BesselY[1,gama r1];
A2=(BesselY[0,beta*r1]*R11/(beta*k2)-
     BesselY[1,beta*r1]*R10)/(BesselY[0,beta*r1]*
BesselJ[1,beta*r1]-
     BesselY[1,beta*r1]*BesselJ[0,beta*r1]);
B2=(BesselJ[0,beta*r1]*R11/(beta*k2)-

BesselJ[1,beta*r1]*R10)/(BesselJ[0,beta*r1]*BesselY[1,beta*r1]-
     BesselJ[1,beta*r1]*BesselY[0,beta*r1]);
R1=A1*BesselJ[0,gama*r]+B1*BesselY[0,gama*r];
R2=A2*BesselJ[0,beta*r]+B2*BesselY[0,beta*r];
DR1=gama*k1*(A1*BesselJ[1,gama r]+B1*BesselY[1,gama r]);
DR2=beta*k2*(A2*BesselJ[1,beta r]+B2*BesselY[1,beta r]);
f=((BesselY[0,beta*r1]*R11/(beta*k2)-
BesselY[1,beta*r1]*R10)*BesselJ[0,beta*r2]-
(BesselJ[0,beta*r1]*R11/(beta*k2)-
BesselJ[1,beta*r1]*R10)*BesselY[0,beta*r2])/(BesselY[0,beta*r1]*Be
sselJ[1,beta*r1]-BesselY[1,beta*r1]*BesselJ[0,beta*r1]);
```

(*Calculating the Eigenvalues*)

```
eig[0]=0;
Do[LL=eig[j-
1]+df;Do[UL=LL+df;Lf=f/.lamb->LL;Uf=f/.lamb->UL;LL=If[(Lf*Uf)>0,UL,
Break[]];If[Uf/Lf>1,df=2*df+df0/5];df=If[df<df0/4,df,df0];If[Uf/Lf
≤1,df=3*df/4+df0/20],{i,1,100}];
```

```
  x0=(UL+LL)/2;dx=(UL-LL)/2;
  Do[UL=(x0+dx);
   LL=(x0-dx);
   ML=(UL+LL)/2;
   f0=f/.lamb->ML;f1=f/.lamb->LL;f2=f/.lamb->UL;
   fp=(f2-f1)/2/dx;
   fpp=(f1+f2-2*f0)/dx^2;
   h=-f0/fp;e=-fpp*h^2/2/(fp+h*fpp);
   x0=x0+h+e;
   dx=Abs[h]/10;If[dx<10^-
10,Break[]];pi=pp,{pp,1,4}];x0=N[Round[x0*10^30]/10^30,30];
  eig[j]=x0;df=df0/100;
  Print["Eigenvalue(",j,")=",N[eig[j],14],"
Eigenfunction=",N[f/.lamb->eig[j]]],{j,1,Nterm}];
```

(*Calculating Temperature and Heat Flux*)

```
 Norm1 =Pi*(A1*r1*BesselJ[0,gama*r1]+B1*r1*BesselY[0,gama*r1])^ 2
+ Pi*(A1*r1*BesselJ[1,gama*r1]+B1*r1*BesselY[1,gama*r1])^2 -
Pi*(A1*r0*BesselJ[1,gama*r0]+B1*r0*BesselY[1,gama*r0])^2 ;
Norm2=Pi*(A2*r2*BesselJ[1,beta*r2]+B2*r2*BesselY[1,beta*r2])^2-
Pi*(A2*r1*BesselJ[0,beta*r1]+B2*r1*BesselY[0,beta*r1])^2-
Pi*(A2*r1*BesselJ[1,beta*r1]+B2*r1*BesselY[1,beta*r1])^2 ;
P1=2*A1*Pi*(r1*BesselJ[1,gama*r1]-
r0*BesselJ[1,gama*r0])/gama+2*B1*Pi*(r1*BesselY[1,gama*r1]-
r0*BesselY[1,gama*r0])/gama;
P2=2*A2*Pi*(r2*BesselJ[1,beta*r2]-
r1*BesselJ[1,beta*r1])/beta+2*B2*Pi*(r2*BesselY[1,beta*r2]-
r1*BesselY[1,beta*r1])/beta;
Temp1=0;Temp2=0;P=Cp1*P1*(Ti1-Tw2)+Cp2*P2*(Ti2-
Tw2);PG=P1*g1+P2*g2;Nrm=Cp1*Norm1+Cp2*Norm2;Q1=0;Q2=0;
Do[lamb=eig[i];
  Tmp1i=R1*P*Exp[-beta^2*tm]/Nrm;Tmp2i=R2*P*Exp[-beta^2*tm]/Nrm;
  Q1i=DR1*P*Exp[-beta^2*tm]/Nrm;Q2i=DR2*P*Exp[-beta^2*tm]/Nrm;
  Tg1=R1*PG*(0-Exp[-beta^2*tm])/beta^2/Nrm;Tg2=R2*PG*(0-Exp[-
beta^2*tm])/beta^2/Nrm;
  Qg1=DR1*PG*(0-Exp[-beta^2*tm])/beta^2/Nrm;Qg2=DR2*PG*(0-Exp[-
beta^2*tm])/beta^2/Nrm;
  Temp1=Temp1+Tmp1i+Tg1;
  Temp2=Temp2+Tmp2i+Tg2;
  Q1=Q1+Q1i+Qg1;
  Q2=Q2+Q2i+Qg2,{i,1,Nterm}];
```

(*Steady State Solutions*)

```
Phi=g2*r2^2*(1-(r1/r2)^2)/(4*k2)-g1*r0^2*(1-(r1/r0)^2)/(4*k1);
Psi=(g1-g2)*r1^2/(2*k1);
D1=(Phi-Psi*Log[r1/r0])/(k2*Log[r1/r0]/k1-Log[r1/r2]);
C1=k2*D1/k1+(g1-g2)*r1^2/(2*k1);
Tsp1=g1*r0^2*(1-(r/r0)^2)/(4*k1)+C1*Log[r/r0];
Tsp2=g2*r2^2*(1-(r/r2)^2)/(4*k2)+D1*Log[r/r2];
Qsp1=g1*r/2-k1*C1/r;
Qsp2=g2*r/2-k2*D1/r;
Temp1=Temp1+Tsp1;Temp2=Temp2+Tsp2;
Q1=Q1+Qsp1;Q2=Q2+Qsp2;
```

(*Printing Selected Temperature Values*)

```
Print["  "];
```

```
 Print["  t   ", " T(",N[r0],",t)"," T(",N[(r0+r1)/2],",t)  ","
T(",N[r1],",t)","  T(",N[(r1+r2)/2],",t)  "," T(",N[r2],",t)"];
t=1/10000;Tm1=Temp1/.tm–>t;Tm2=Temp2/.tm–>t;Print[N[t,2],"
",Chop[N[Tm1/.r–>r0,10]],"   ",N[Tm1/.r–>(r0+r1)/2,10],"
",N[Tm1/.r–>r1,10]," ",N[Tm2/.r–>(r1+r2)/2,10],"
",Chop[N[Tm2/.r–>r2,10]]];
t=1/1000;Tm1=Temp1/.tm–>t;Tm2=Temp2/.tm–>t;Print[N[t,2],"
",Chop[N[Tm1/.r–>r0,10]],"   ",N[Tm1/.r–>(r0+r1)/2,10],"
",N[Tm1/.r–>r1,10]," ",N[Tm2/.r–>(r1+r2)/2,10],"
",Chop[N[Tm2/.r–>r2,10]]];
t=1/200;Tm1=Temp1/.tm–>t;Tm2=Temp2/.tm–>t;Print[N[t,2],"
",Chop[N[Tm1/.r–>r0,10]],"   ",N[Tm1/.r–>(r0+r1)/2,10],"
",N[Tm1/.r–>r1,10]," ",N[Tm2/.r–>(r1+r2)/2,10],"
",Chop[N[Tm2/.r–>r2,10]]];
t=1/100;Tm1=Temp1/.tm–>t;Tm2=Temp2/.tm–>t;Print[N[t,2],"
",Chop[N[Tm1/.r–>r0,10]],"   ",N[Tm1/.r–>(r0+r1)/2,10],"
",N[Tm1/.r–>r1,10]," ",N[Tm2/.r–>(r1+r2)/2,10],"
",Chop[N[Tm2/.r–>r2,10]]];Do[t=n/40;Tm1=Temp1/.tm–>t;Tm2=Temp2/.tm–
>t;Print[N[t,2],"  ",Chop[N[Tm1/.r–>r0,10]],"
",N[Tm1/.r–>(r0+r1)/2,10]," ",N[Tm1/.r–>r1,10],"
",N[Tm2/.r–>(r1+r2)/2,10],"  ",Chop[N[Tm2/.r–>r2,10]]],{n,1,16}];
t=1/2;Tm1=Temp1/.tm–>t;Tm2=Temp2/.tm–>t;Print[N[t,2],"
",Chop[N[Tm1/.r–>r0,10]],"   ",N[Tm1/.r–>(r0+r1)/2,10],"
",N[Tm1/.r–>r1,10]," ",N[Tm2/.r–>(r1+r2)/2,10],"
",Chop[N[Tm2/.r–>r2,10]]];
```

(*Printing Selected Heat Flux Values*)

```
Print["  "];
 Print["  t   ", "    q(",N[r0],",t)","   q(",N[(r0+r1)/2],",t)
","  q(",N[r1],",t)","   q(",N[(r1+r2)/2],",t)  ","
q(",N[r2],",t)"];
t=1/10000;Qm1=Q1/.tm–>t;Qm2=Q2/.tm–>t;Print[N[t,2],"
",N[Qm1/.r–>r0,10]," ",N[Qm1/.r–>(r0+r1)/2,10],"
",N[Qm1/.r–>r1,10]," ",N[Qm2/.r–>(r1+r2)/2,10],"
",Chop[N[Qm2/.r–>r2,10]]];
t=1/1000;Qm1=Q1/.tm–>t;Qm2=Q2/.tm–>t;Print[N[t,2],"
",Chop[N[Qm1/.r–>r0,10]]," ",N[Qm1/.r–>(r0+r1)/2,10],"
",N[Qm1/.r–>r1,10]," ",N[Qm2/.r–>(r1+r2)/2,10],"
",Chop[N[Qm2/.r–>r2,10]]];
t=1/200;Qm1=Q1/.tm–>t;Qm2=Q2/.tm–>t;Print[N[t,2],"
",Chop[N[Qm1/.r–>r0,10]]," ",N[Qm1/.r–>(r0+r1)/2,10],"
",N[Qm1/.r–>r1,10]," ",N[Qm2/.r–>(r1+r2)/2,10],"
",Chop[N[Qm2/.r–>r2,10]]];
t=1/100;Qm1=Q1/.tm–>t;Qm2=Q2/.tm–>t;Print[N[t,2],"
",Chop[N[Qm1/.r–>r0,10]]," ",N[Qm1/.r–>(r0+r1)/2,10],"
",N[Qm1/.r–>r1,10]," ",N[Qm2/.r–>(r1+r2)/2,10],"
",Chop[N[Qm2/.r–>r2,10]]];Do[t=n/40;Qm1=Q1/.tm–>t;Qm2=Q2/.tm–>t;Pri
nt[N[t,2],"  ",Chop[N[Qm1/.r–>r0,10]]," ",N[Qm1/.r–>(r0+r1)/2,10],"
```

```
",N[Qm1/.r->r1,10]," ",N[Qm2/.r->(r1+r2)/2,10],"
",Chop[N[Qm2/.r->r2,10]]],{n,1,16}];
t=1/2;Qm1=Q1/.tm->t;Qm2=Q2/.tm->t;Print[N[t,2],"
",Chop[N[Qm1/.r->r0,10]]," ",N[Qm1/.r->(r0+r1)/2,10],"
",N[Qm1/.r->r1,10]," ",N[Qm2/.r->(r1+r2)/2,10],"
",Chop[N[Qm2/.r->r2,10]]];
```