R21B01T0

Satish Nallapaneni & James V. Beck – June 26, 2014

Hollow cylinder insulated at inner radius and heating through a step change in temperature at outer radius[1]

Contents

R21B01T0

Satish Nallapaneni & James V. Beck – June 26, 2014

1. Problem description

This problem is for a homogeneous annulus of inner radius R_1 and the outer radius R_2. It is subjected to heating by step change in temperature T_0 at R_2. The inner surface is insulated. At time t=0 temperature at every point inside the cylinder is 0.

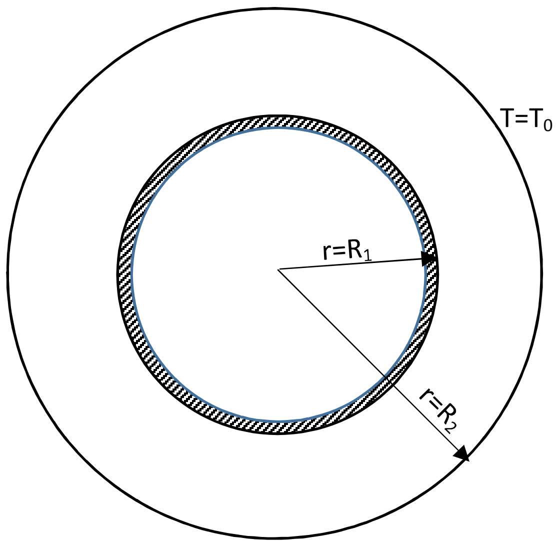

Figure 1. Schematic of R21B01T0 problem

2. Dimensional R21B01T0 problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t}, R_1<r<R_2, t>0 \quad \text{(R21B01T0-1)}$$

$$k\frac{\partial T}{\partial r}(R_1,t)=0 \quad \text{(R21B01T0-2)}$$

$$T(R_2,t)=T_0 \quad \text{(R21B01T0-3)}$$

$$T(r,0)=0 \quad \text{(R21B01T0-4)}$$

3. Dimensional Solution

The dimensional temperature solution is given as

$$T_{\text{R21B01T0}}(r,t)=T_0-\pi T_0\sum_{m=1}^{\infty}e^{\frac{-\beta_m^2\alpha t}{R_1^2}}\frac{J_1^2(\beta_m)\left[J_0\left(\beta_m\frac{r}{R_1}\right)Y_0\left(\beta_m\frac{R_2}{R_1}\right)-J_0\left(\beta_m\frac{R_2}{R_1}\right)Y_0\left(\beta_m\frac{r}{R_1}\right)\right]}{J_1^2(\beta_m)-J_0^2\left(\beta_m\frac{R_2}{R_1}\right)} \quad \text{(R21B01T0-5)}$$

The eigenvalues are found from the eigencondition

$$J_1(\beta_m)Y_0\left(\beta_m \frac{R_2}{R_1}\right) - J_0\left(\beta_m \frac{R_2}{R_1}\right)Y_1(\beta_m) = 0 \qquad \text{(R21B01T0-6)}$$

Appendix A contains a Matlab program for finding the eigenvalues. Appendix B contains a Matlab program for the solution of the temperature and heat flux. Appendix C is for plots of the temperature and heat flux.

4. Dimensionless groups

$$\tilde{T}(\tilde{r},\tilde{t}) = \frac{T(r,t)}{T_0},\ \tilde{q} = \frac{q(r,t)}{kT_0 / R_1},\ \tilde{r} = \frac{r}{R_1},\ \tilde{t} = \frac{\alpha t}{R_1^2},\ \tilde{R} = \frac{R_2}{R_1} \qquad \text{(R21B01T0-7)}$$

5. Dimensionless R21B01T0 problem

$$\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{T}}{\partial \tilde{r}}\right) = \frac{\partial \tilde{T}}{\partial \tilde{t}}, \qquad 1 < r < \tilde{R}, \qquad \tilde{t} > 0 \qquad \text{(R21B01T0-8)}$$

$$\frac{\partial \tilde{T}}{\partial \tilde{r}}(1,\tilde{t}) = 0 \qquad \text{(R21B01T0-9)}$$

$$\tilde{T}(\tilde{R},\tilde{t}) = T_0 \qquad \text{(R21B01T0-10)}$$

$$\tilde{T}(\tilde{r},0) = 0 \qquad \text{(R21B01T0-11)}$$

6. Dimensionless Solution

The transient problem(R21B01T0-8) is solved by using Green's function.

$$\tilde{T}_{\text{R21B01T0}}(\tilde{r},\tilde{t}) = 1 - \pi \sum_{m=1}^{\infty} e^{-\beta_m^2 \tilde{t}} \frac{J_1^2(\beta_m)\begin{bmatrix} J_0(\beta_m \tilde{r})Y_0(\beta_m \tilde{R}) \\ -J_0(\beta_m \tilde{R})Y_0(\beta_m \tilde{r}) \end{bmatrix}}{J_1^2(\beta_m) - J_0^2(\beta_m \tilde{R})} \qquad \text{(R21B01T0-12)}$$

This equation can also be written as

$$\tilde{T}_{\text{R21B01T0}}(\tilde{r},\tilde{t}) = 1 - \pi \sum_{m=1}^{\infty} e^{-\beta_m^2 \tilde{t}} \frac{J_0^2(\beta_m) N_m(\tilde{r},\tilde{R})}{J_1^2(\beta_m) - J_0^2(\beta_m \tilde{R})} \qquad \text{(R21B01T0-13)}$$

where,

$$N_m\left(\tilde{r},\tilde{R}\right)=J_0\left(\beta_m\tilde{r}\right)Y_0\left(\beta_m\tilde{R}\right)-J_0\left(\beta_m\tilde{R}\right)Y_0\left(\beta_m\tilde{r}\right) \qquad \text{(R21B01T0-14)}$$

The non-dimensional heat flux is given by the negative of the first derivative of eq. (R21B01T0-12) with respect to $\tilde{r}$ which produces

$$\tilde{q}_{\text{R21B01T0}}(\tilde{r},\tilde{t})=-\pi\sum_{m=1}^{\infty}e^{-\beta_m^2\tilde{t}}\frac{\beta_m J_0^2\left(\beta_m\right)}{J_1^2\left(\beta_m\right)-J_0^2\left(\beta_m\tilde{R}\right)}\begin{pmatrix}J_1\left(\beta_m\tilde{r}\right)Y_0\left(\beta_m\tilde{R}\right)\\-J_0\left(\beta_m\tilde{R}\right)Y_1\left(\beta_m\tilde{r}\right)\end{pmatrix} \qquad \text{(R21B01T0-15)}$$

7. Dimensionless eigenvalues

The dimensionless eigenvalues are found from the eigencondition

$$J_1\left(\beta_m\right)Y_0\left(\beta_m\tilde{R}\right)-J_0\left(\beta_m\tilde{R}\right)Y_1\left(\beta_m\right)=0 \qquad \text{(R21B01T0-16)}$$

Notice that the eigenvalues found from (R21B01T0-16) are no different from those found from(R21B01T0-6).

An efficient iteration procedure for finding the eigenvalues is contained in the Matlab function feigR21(num, R) in Appendix A. This function uses a complicated expression as the initial estimate of eigenvalues as shown below. [2]

```
beta = (j-1/2)*pi()/x + ((0.31+0.001*x)/(1+0.5*x)-
0.12*(0.2*x^1)/(x^2+1.4)+0.067*x/(x^2+1))/j^1.667+(j-1)*2*0.03/j^2;
```

The eigencondition differs only if the type of the boundary condition differs. It is independent of the source of heating. Hence the eigenvalues for R21B10T0 and R21B01T0 will be the same.

Maximum number of terms required in the summation is to get a solution accurate up to 10^{-A} is obtained by rounding off the above expression to the nearest integer.

$$m_{\max}=floor\left(\frac{1}{2}+\left(\frac{\tilde{R}-1}{\pi}\sqrt{\frac{A\ln(10)}{\tilde{t}}}\right)\right) \qquad \text{(R21B01T0-17)}$$

The number of terms increases linearly with the aspect ratio $\tilde{R}$ and inversely with the square root of time. As $\tilde{R}$ increases, the number of terms indicated by this equation becomes large and can even go to infinity. Notice that the number of terms is independent of location $\tilde{r}$.

8. Discussion

The variation of temperature and heat flux with respect to time and position is shown in Tables 1 to 4 in Appendix C.

Intrinsic Verification:

One way to verify the correctness of the solution is to check if the boundary conditions are satisfied. Tables 1 and 3 show that the boundary condition at the outer surface is readily satisfied. Similarly, Tables 2 and 4 show that the boundary condition at the inner surface is also satisfied. Remember that these values are not manually placed there but are calculated values.

An even more powerful way to verify the accuracy of the solution is to use knowledge of the penetration time; that is, at very small times, the effect of heating a surface is not felt at locations away from the heating surface. For these small times and away from the heated surface, the temperatures must be zero. These are not manually placed zeroes but computed zeroes. See the first time entry in Tables 1, 2, 3 and 4.

Prof. de Monte found out the expression to calculate the penetration time in Cartesian coordinate system. [3, eq. (17)] Based on his work, the penetration time for this problem can be calculated using the equation

$$\frac{\alpha t_p}{(R_2 - r)^2} = \frac{1}{10A} \qquad \text{(R21B01T0-18)}$$

Now, let's look at how the concept of penetration times can be used as a way of intrinsic verification to check the accuracy of the solution. The dimensionless penetration time for a location of $\tilde{r} = 1.75$ is found out to be $\tilde{t}_p = 0.000625$ for $\tilde{R} = 2$ and 10-digit accuracy. [5, eq. (21)] In other words, the effect of the heated surface is not felt in the region $\tilde{r} < 1.75$ during $\tilde{t} < 0.000625$. This can be verified through the following table.

$\tilde{R}$	$\tilde{t}$	$\tilde{T}(1,\tilde{t})$	$\tilde{T}(1.25,\tilde{t})$	$\tilde{T}(1.5,\tilde{t})$	$\tilde{T}(1.75,\tilde{t})$	$\tilde{T}(\tilde{R},\tilde{t})$
2.0000	0.00010	0.0000000000	0.0000000000	0.0000000000	0.0000000000	1.0000000000
2.0000	0.00020	0.0000000000	0.0000000000	0.0000000000	0.0000000000	1.0000000000
2.0000	0.00030	0.0000000000	0.0000000000	0.0000000000	0.0000000000	1.0000000000
2.0000	0.00040	0.0000000000	0.0000000000	0.0000000000	0.0000000000	1.0000000000
2.0000	0.00050	0.0000000000	0.0000000000	0.0000000000	0.0000000000	1.0000000000

2.0000	0.00060	0.0000000000	0.0000000000	0.0000000000	0.0000000000	1.0000000000
$\tilde{R}$	$\tilde{t}$	$\tilde{q}(1,\tilde{t})$	$\tilde{q}(1.25,\tilde{t})$	$\tilde{q}(1.5,\tilde{t})$	$\tilde{q}(1.75,\tilde{t})$	$\tilde{q}(\tilde{R},\tilde{t})$
2.0000	0.00010	0.0000000000	0.0000000000	0.0000000000	0.0000000000	56.1686041645
2.0000	0.00020	0.0000000000	0.0000000000	0.0000000000	0.0000000000	39.6437262111
2.0000	0.00030	0.0000000000	0.0000000000	0.0000000000	0.0000000000	32.3228853046
2.0000	0.00040	0.0000000000	0.0000000000	0.0000000000	0.0000000000	27.9587676159
2.0000	0.00050	0.0000000000	0.0000000000	0.0000000000	0.0000000000	24.9805288245
2.0000	0.00060	0.0000000000	0.0000000000	0.0000000000	0.0000000001	22.7820700504

9. References

1. Cole, K.D., Beck, J.V., Haji-Sheikh, A. and Litkouhi, B., Heat Conduction Using Green's Functions, CRC Press, 2011, 2nd Edition.
2. The formulation used for calculating the eigenvalues in feigR21(num,R) is provided by Prof. Haji-Sheikh, University of Texas, Arlington.
3. de Monte, F., Beck, J. V., and Amos, D. E., Diffusion of thermal disturbances in two-dimensional Cartesian transient heat conduction, Int. J. Heat Mass Transfer, Vol. 51, No. 25-26, pp. 5931-5941, December 2008.
4. Cole, K.D., Beck, J.V., Hollow Cylinder, zero heat flux at outer boundary and temperature jump at inner boundary, Exact Analytical Conduction Toolbox, exact.unl.edu, Nov 7, 2013.
5. Satish Nallapaneni and James V. Beck, Hollow cylinder with heating through constant heat flux at outer radius and zero temperature at inner radius, Exact Analytical Conduction Toolbox, exact.unl.edu, June 18, 2014.
6. Satish Nallapaneni and James V. Beck, Hollow cylinder with constant heat flux at inner radius and constant zero temperature at outer radius, Exact Analytical Conduction Toolbox, exact.unl.edu, April 9, 2014.

Appendix A. MATLAB function feigR12(num,R,Bi) for finding the eigenvalues of R12 case.

```
%feigR21.m
%March 31, 2014.  R is the ratio R2/R1, num =number of eigenvalues
% R must  be greater than 1
%Author: Satish Nallapaneni
%Eigen condition: f=besselj(1,bet)*bessely(0,bet*R)-
bessely(1,bet)*besselj(0,bet*R);
function BB=feigR21(num,R)
    x=R-1;
for j=1:num
    beta = (j-1/2)*pi()/x + ((0.31+0.001*x)/(1+0.5*x)-
0.12*(0.2*x^1)/(x^2+1.4)+0.067*x/(x^2+1))/j^1.667+(j-1)*2*0.03/j^2;
    x0=beta; dx=pi()/x/20;
    for pp=1:6
        UL=(x0+dx);
        LL=(x0-dx);
        ML=(UL+LL)/2;
        f0=besselj(1,ML)*bessely(0,ML*R)-bessely(1,ML)*besselj(0,ML*R);
        f1=besselj(1,LL)*bessely(0,LL*R)-bessely(1,LL)*besselj(0,LL*R);
        f2=besselj(1,UL)*bessely(0,UL*R)-bessely(1,UL)*besselj(0,UL*R);
        fp=(f2-f1)/2/dx;
        fpp=(f1+f2-2*f0)/dx^2;
        h=-f0/fp;
        eps=-fpp*h^2/2/(fp+h*fpp);
        x0=x0+h+eps;        %Newton Raphson method for finding eigenvalues
        dx=h^2;
        if dx<10^-10
            break
        end
    end
    eig(j) = x0;
end
BB=eig;
```

Appendix B. MATLAB function fdR21B01T0.

- **fdR21B01T0**

  Heat conduction function for the R21B01T0 case.

- **Syntax**

```
[Td,qd] = fdR21B01T0(rv,tv,R,A)
```

- **Description**

fdR21B01T0(rv,tv,R,A) returns the dimensionless temperature *Td* and heat flux *qd* solutions at a given dimensionless location $\tilde{r}$ from the inner surface, between 1 and $\tilde{R}$, and at a given dimensionless time $\tilde{t}$, with an accuracy of 10^{-A} (*A = 2,3,…,15*), for the R21B01T0 problem.

If *rv* and *tv* are not single values but arrays (length(*rv*) = *n* and length(*tv*) = *m*) defining the dimensionless locations and times of interest, respectively, the above function returns the dimensionless temperature *Td* and heat flus *qd* as double scripted arrays, where size (*Td*) = size(*qd*)=[*m, n*].

- **Examples**

Example 1

```
>> [Td, qd] = fdR21B01T0(1.5,0.1,2,15)

Td =

   0.306775212579212


qd =

   0.996760738743538
```

Example 2

```
A =

    15
```

```
>> rv=[1.15 1.5 1.75]'

rv =

   1.150000000000000
   1.500000000000000
   1.750000000000000

>> tv=[0.01 0.1 0.3]'

tv =

   0.010000000000000
   0.100000000000000
   0.300000000000000

>> R=2

R =

     2

>> [Td, qd] = fdR21B01T0(rv,tv,R,A)

Td =

   0.000000002443030   0.000470260066872   0.082466212110522
   0.088019143761261   0.306775212579212   0.617792320795009
   0.498533333626958   0.651889024618548   0.820408942119899


qd =

   0.000000105497427   0.012428635174915   1.241259402634826
   0.276257203953171   0.996760738743538   1.452408306110740
   0.230061614975904   0.606386444812265   0.718924711985278
```

R21B01T0

Satish Nallapaneni & James V. Beck – June 26, 2014


```
% fdR21B10T0 function
% Author: Satish Nallapaneni
% Revision June 27, 2014
% INPUTS:
% R: radius ratio R2/R1
% rv: dimensionless location starting at rd=r/R1=1 and ending at rd=R2/R1=R
% tv: dimensionless time starting at td=0
% A = desired accuracy (1E-A =10^-A); A=2,3, ..., 15
% OUTPUTS:
% Td: dimensionless temperature calculated at (xd,td) to desired accuracy A
% qd: dimensionless heat flux calculated at (xd,td) to desired accuracy A
function [Td,qd]=fdR21B01T0(rv,tv,R,A)
   srv=length(rv);
   stv=length(tv);
   Temp=zeros(stv,srv);     % Preallocating arrays for speed
   flux=zeros(stv,srv);     % Preallocating arrays for speed
 % calculate number of eigenvalues required to obtain solution with accuracy
A
   mmax1=floor(1.2*(0.5+(R-1)/pi*sqrt(A*log(10)/min(tv))));
   bet=feigR21(mmax1,R);% Call the function to get eigenvalues
for ir = 1:srv     % begin space loop
    r=rv(ir);
    term1=1;    % steady-state temperature solution
    term2=0;         % steady-state heat flux solution
    for it=1:stv       % begin time loop
        t=tv(it);
    % calculate m_max for every timestep as it reduces computational cost
        mmax=floor(1.2*(0.5+(R-1)/pi*sqrt(A*log(10)/t)));
        term3 = 0;
        term4 = 0;
        for ii=1:mmax
            bt=bet(ii);
            Nr = (besselj(0,bt*r)*bessely(0,bt*R)-
besselj(0,bt*R)*bessely(0,bt*r));
            Denominator = besselj(1,bt)*besselj(1,bt)-
besselj(0,bt*R)*besselj(0,bt*R);
            Norm = Denominator/(besselj(1,bt)*besselj(1,bt));
            term3 = term3 - (pi)*exp(-bt*bt*t)*Nr/(Norm);
            Nm = (besselj(1,bt*r)*bessely(0,bt*R)-
besselj(0,bt*R)*bessely(1,bt*r));
            term4 = term4 - (pi)*exp(-bt*bt*t)*bt*Nm/(Norm);
        end
        Temp(it,ir) = term1 + term3;  % total temperature solution
        flux(it,ir) = term2 + term4;  % total heat flux solution
    end
end
Td=abs(Temp); qd=abs(flux);
```

Appendix C. Tables and plots of dimensionless temperatures and heat fluxes

Table 1. Temperatures as a function of time and position for R21B01T0 case for R_2/R_1 = 1.1.

$\tilde{R}$	$\tilde{t}$	$\tilde{T}(1,\tilde{t})$	$\tilde{T}(1.25,\tilde{t})$	$\tilde{T}(1.5,\tilde{t})$	$\tilde{T}(1.75,\tilde{t})$	$\tilde{T}(\tilde{R},\tilde{t})$
1.1000	0.00001	0.0000000000	0.0000000000	0.0000000000	0.0000000229	1.0000000000
1.1000	0.00005	0.0000000000	0.0000000000	0.0000005868	0.0125630189	1.0000000000
1.1000	0.00010	0.0000000000	0.0000001178	0.0004165366	0.0779924050	1.0000000000
1.1000	0.00020	0.0000012004	0.0001831976	0.0127120367	0.2137478493	1.0000000000
1.1000	0.00030	0.0000932163	0.0022791762	0.0421990875	0.3109990335	1.0000000000
1.1000	0.00040	0.0008506978	0.0083086184	0.0789192460	0.3811306954	1.0000000000
1.1000	0.00050	0.0032698701	0.0184234783	0.1165361466	0.4341782520	1.0000000000
1.1000	0.00060	0.0081247415	0.0317944744	0.1524471172	0.4759521065	1.0000000000
1.1000	0.00070	0.0156989224	0.0475016042	0.1857988168	0.5098966753	1.0000000000
1.1000	0.00080	0.0258876869	0.0648079434	0.2164740658	0.5381686761	1.0000000000
1.1000	0.00090	0.0383752937	0.0831743536	0.2446477902	0.5621909824	1.0000000000
1.1000	0.00100	0.0527678437	0.1022140227	0.2705936565	0.5829458263	1.0000000000
1.1000	0.00200	0.2356904079	0.2928807116	0.4570551088	0.7059659255	1.0000000000
1.1000	0.00300	0.4051440497	0.4514956063	0.5821943265	0.7755118374	1.0000000000
1.1000	0.00400	0.5393161790	0.5754097996	0.6769412969	0.8266115194	1.0000000000
1.1000	0.00500	0.6434698053	0.6714243463	0.7500346620	0.8658620173	1.0000000000
1.1000	0.00600	0.7241021409	0.7457367967	0.8065723267	0.8962038355	1.0000000000
1.1000	0.00700	0.7865016108	0.8032434280	0.8503202507	0.9196798602	1.0000000000
1.1000	0.00800	0.8347885748	0.8477439189	0.8841734109	0.9378459408	1.0000000000
1.1000	0.00900	0.8721545013	0.8821797335	0.9103699588	0.9519033531	1.0000000000
1.1000	0.01000	0.9010693665	0.9088271884	0.9306416202	0.9627813907	1.0000000000
1.1000	0.02000	0.9923829948	0.9929802958	0.9946598630	0.9971344130	1.0000000000
1.1000	0.03000	0.9994135409	0.9994595292	0.9995888447	0.9997793687	1.0000000000
1.1000	0.04000	0.9999548465	0.9999583873	0.9999683438	0.9999830128	1.0000000000
1.1000	0.05000	0.9999965235	0.9999967961	0.9999975627	0.9999986921	1.0000000000
1.1000	0.06000	0.9999997323	0.9999997533	0.9999998123	0.9999998993	1.0000000000
1.1000	0.07000	0.9999999794	0.9999999810	0.9999999856	0.9999999922	1.0000000000
1.1000	0.08000	0.9999999984	0.9999999985	0.9999999989	0.9999999994	1.0000000000
1.1000	0.09000	0.9999999999	0.9999999999	0.9999999999	1.0000000000	1.0000000000
1.1000	0.10000	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
1.1000	steady state	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000

Table 2. Heat Fluxes as a function of time and position for R21B01T0 case for R_2/R_1 = 1.1.

$\tilde{R}$	$\tilde{t}$	$\tilde{q}(1,\tilde{t})$	$\tilde{q}(1.25,\tilde{t})$	$\tilde{q}(1.5,\tilde{t})$	$\tilde{q}(1.75,\tilde{t})$	$\tilde{q}(\tilde{R},\tilde{t})$
1.1000	0.00001	0.0000000000	0.0000000000	0.0000000001	0.0000295400	177.9574965976
1.1000	0.00005	0.0000000000	0.0000000000	0.0003040649	3.5403749441	79.3330816332
1.1000	0.00010	0.0000000000	0.0000455990	0.1112809931	11.9266616091	55.9632377262
1.1000	0.00020	0.0000000000	0.0364387064	1.7880852491	18.3769997090	39.4380149919
1.1000	0.00030	0.0000000000	0.3096041847	4.1314092898	19.4289322888	32.1169076129
1.1000	0.00040	0.0000000000	0.8631270530	6.0149031572	19.1312637056	27.7525639708
1.1000	0.00050	0.0000000000	1.5503622223	7.3436416643	18.4714048128	24.7741250086
1.1000	0.00060	0.0000000000	2.2393661727	8.2447940564	17.7362049391	22.5754820031
1.1000	0.00070	0.0000000000	2.8577672833	8.8426452027	17.0180723069	20.8666381690
1.1000	0.00080	0.0000000000	3.3769851769	9.2271289561	16.3460436214	19.4890464810
1.1000	0.00090	0.0000000000	3.7929898424	9.4594729344	15.7263975373	18.3476322130
1.1000	0.00100	0.0000000000	4.1135432495	9.5813297716	15.1569849843	17.3813073819
1.1000	0.00200	0.0000000000	4.5220618042	8.4670485571	11.1783048443	11.9887393555
1.1000	0.00300	0.0000000000	3.6464340045	6.6627631375	8.5896208053	9.1235433866
1.1000	0.00400	0.0000000000	2.8375391622	5.1677378376	6.6403994244	7.0435550899
1.1000	0.00500	0.0000000000	2.1974722916	4.0002270163	5.1378387006	5.4487366677
1.1000	0.00600	0.0000000000	1.7006519065	3.0956315402	3.9757366072	4.2162030280
1.1000	0.00700	0.0000000000	1.3160344538	2.3955061305	3.0765346502	3.2626022593
1.1000	0.00800	0.0000000000	1.0183885454	1.8537151728	2.3807127541	2.5246960445
1.1000	0.00900	0.0000000000	0.7880594424	1.4344598680	1.8422659393	1.9536843298
1.1000	0.01000	0.0000000000	0.6098237589	1.1100275449	1.4255999315	1.5118187662
1.1000	0.02000	0.0000000000	0.0469524009	0.0854647877	0.1097617763	0.1164000568
1.1000	0.03000	0.0000000000	0.0036150247	0.0065802240	0.0084509316	0.0089620354
1.1000	0.04000	0.0000000000	0.0002783330	0.0005066338	0.0006506659	0.0006900175
1.1000	0.05000	0.0000000000	0.0000214298	0.0000390075	0.0000500970	0.0000531268
1.1000	0.06000	0.0000000000	0.0000016500	0.0000030033	0.0000038571	0.0000040904
1.1000	0.07000	0.0000000000	0.0000001270	0.0000002312	0.0000002970	0.0000003149
1.1000	0.08000	0.0000000000	0.0000000098	0.0000000178	0.0000000229	0.0000000242
1.1000	0.09000	0.0000000000	0.0000000008	0.0000000014	0.0000000018	0.0000000019
1.1000	0.10000	0.0000000000	0.0000000001	0.0000000001	0.0000000001	0.0000000001
1.1000	steady state	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000

Table 3. Temperatures as a function of time and position for R21B01T0 case for R_2/R_1 = 2.

$\tilde{R}$	$\tilde{t}$	$\tilde{T}(1,\tilde{t})$	$\tilde{T}(1.25,\tilde{t})$	$\tilde{T}(1.5,\tilde{t})$	$\tilde{T}(1.75,\tilde{t})$	$\tilde{T}(\tilde{R},\tilde{t})$
2.0000	0.00001	0.0000000000	0.0000000000	0.0000000000	0.0000000000	1.0000000000
2.0000	0.00010	0.0000000000	0.0000000000	0.0000000000	0.0000000000	1.0000000000
2.0000	0.00100	0.0000000000	0.0000000000	0.0000000000	0.0000000243	1.0000000000
2.0000	0.00200	0.0000000000	0.0000000000	0.0000000000	0.0000825696	1.0000000000
2.0000	0.00300	0.0000000000	0.0000000000	0.0000000001	0.0013353064	1.0000000000
2.0000	0.00400	0.0000000000	0.0000000000	0.0000000262	0.0055481941	1.0000000000
2.0000	0.00500	0.0000000000	0.0000000000	0.0000006623	0.0132807122	1.0000000000
2.0000	0.00600	0.0000000000	0.0000000000	0.0000057881	0.0240391478	1.0000000000
2.0000	0.00700	0.0000000000	0.0000000003	0.0000275110	0.0370146476	1.0000000000
2.0000	0.00800	0.0000000000	0.0000000039	0.0000892283	0.0514507249	1.0000000000
2.0000	0.00900	0.0000000000	0.0000000287	0.0002240969	0.0667482823	1.0000000000
2.0000	0.01000	0.0000000000	0.0000001440	0.0004702601	0.0824662121	1.0000000000
2.0000	0.02000	0.0000015957	0.0002240922	0.0143606158	0.2260912112	1.0000000000
2.0000	0.03000	0.0001231218	0.0027902165	0.0476988946	0.3290597953	1.0000000000
2.0000	0.04000	0.0011168859	0.0101789934	0.0892520205	0.4033754256	1.0000000000
2.0000	0.05000	0.0042686802	0.0225833380	0.1318598376	0.4596357743	1.0000000000
2.0000	0.06000	0.0105492680	0.0389844392	0.1725743462	0.5039789121	1.0000000000
2.0000	0.07000	0.0202785618	0.0582406918	0.2104224765	0.5400441852	1.0000000000
2.0000	0.08000	0.0332742164	0.0794276039	0.2452620726	0.5701104666	1.0000000000
2.0000	0.09000	0.0490903802	0.1018611868	0.2772802452	0.5956808728	1.0000000000
2.0000	0.10000	0.0671921726	0.1250460292	0.3067752126	0.6177923208	1.0000000000
2.0000	0.20000	0.2885957043	0.3511612077	0.5170048342	0.7487276709	1.0000000000
2.0000	0.30000	0.4808022534	0.5282690770	0.6518890246	0.8204089421	1.0000000000
2.0000	0.40000	0.6233254969	0.6579411467	0.7478823352	0.8700832466	1.0000000000
2.0000	0.50000	0.7269476154	0.7520583967	0.8172824753	0.9058603613	1.0000000000
2.0000	0.60000	0.8020857239	0.8202883354	0.8675666961	0.9317692773	1.0000000000
2.0000	0.70000	0.8565496578	0.8697432782	0.9040113826	0.9505460280	1.0000000000
2.0000	0.80000	0.8960259080	0.9055887798	0.9304266338	0.9641553555	1.0000000000
2.0000	0.90000	0.9246386694	0.9315699242	0.9495726204	0.9740194921	1.0000000000
2.0000	1.00000	0.9453774511	0.9504012852	0.9634497964	0.9811691019	1.0000000000
2.0000	steady state	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000

Table 4. Heat Fluxes as a function of time and position for R21B01T0 case for R_2/R_1 = 2.

$\tilde{R}$	$\tilde{t}$	$\tilde{q}(1,\tilde{t})$	$\tilde{q}(1.25,\tilde{t})$	$\tilde{q}(1.5,\tilde{t})$	$\tilde{q}(1.75,\tilde{t})$	$\tilde{q}(\tilde{R},\tilde{t})$
2.0000	0.00001	0.0000000000	0.0000000000	0.0000000000	0.0000000000	178.1622999510
2.0000	0.00010	0.0000000000	0.0000000000	0.0000000000	0.0000000000	56.1686041645
2.0000	0.00100	0.0000000000	0.0000000000	0.0000000000	0.0000031163	17.5901101621
2.0000	0.00200	0.0000000000	0.0000000000	0.0000000000	0.0054344350	12.3640535546
2.0000	0.00300	0.0000000000	0.0000000000	0.0000000106	0.0598728669	10.0486655727
2.0000	0.00400	0.0000000000	0.0000000000	0.0000016784	0.1902940931	8.6683254957
2.0000	0.00500	0.0000000000	0.0000000000	0.0000341268	0.3710792014	7.7262706792
2.0000	0.00600	0.0000000000	0.0000000006	0.0002499057	0.5692437445	7.0308265719
2.0000	0.00700	0.0000000000	0.0000000160	0.0010234583	0.7632044511	6.4902900097
2.0000	0.00800	0.0000000000	0.0000001839	0.0029192984	0.9420799603	6.0545453819
2.0000	0.00900	0.0000000000	0.0000012213	0.0065493650	1.1016494223	5.6935861337
2.0000	0.01000	0.0000000000	0.0000055223	0.0124286352	1.2412594026	5.3882035292
2.0000	0.02000	0.0000000000	0.0043781515	0.1978697242	1.8892035216	3.7340946964
2.0000	0.03000	0.0000000000	0.0369148011	0.4532280445	1.9758553226	3.0007194671
2.0000	0.04000	0.0000000000	0.1021439483	0.6544121917	1.9264515566	2.5631850231
2.0000	0.05000	0.0000000000	0.1821278849	0.7926400587	1.8429341517	2.2643445207
2.0000	0.06000	0.0000000000	0.2611690375	0.8830723740	1.7542135796	2.0435558934
2.0000	0.07000	0.0000000000	0.3309140255	0.9400260616	1.6692210030	1.8718010459
2.0000	0.08000	0.0000000000	0.3882763188	0.9737289200	1.5905036796	1.7332137027
2.0000	0.09000	0.0000000000	0.4330556473	0.9910857401	1.5183849967	1.6182794628
2.0000	0.10000	0.0000000000	0.4663925350	0.9967607387	1.4524083061	1.5208865245
2.0000	0.20000	0.0000000000	0.4791120620	0.8230866723	0.9993637459	0.9770336044
2.0000	0.30000	0.0000000000	0.3616381985	0.6063864448	0.7189247120	0.6959435127
2.0000	0.40000	0.0000000000	0.2635516107	0.4404910298	0.5205422432	0.5032086795
2.0000	0.50000	0.0000000000	0.1911673272	0.3193690871	0.3772396678	0.3646087215
2.0000	0.60000	0.0000000000	0.1385739772	0.2314911972	0.2734212255	0.2642594857
2.0000	0.70000	0.0000000000	0.1004410437	0.1677878020	0.1981775495	0.1915363710
2.0000	0.80000	0.0000000000	0.0728006843	0.1216141561	0.1436407599	0.1388271092
2.0000	0.90000	0.0000000000	0.0527665839	0.0881470087	0.1041120680	0.1006230854
2.0000	1.00000	0.0000000000	0.0382456808	0.0638897203	0.0754613328	0.0729324871
2.0000	steady state	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000

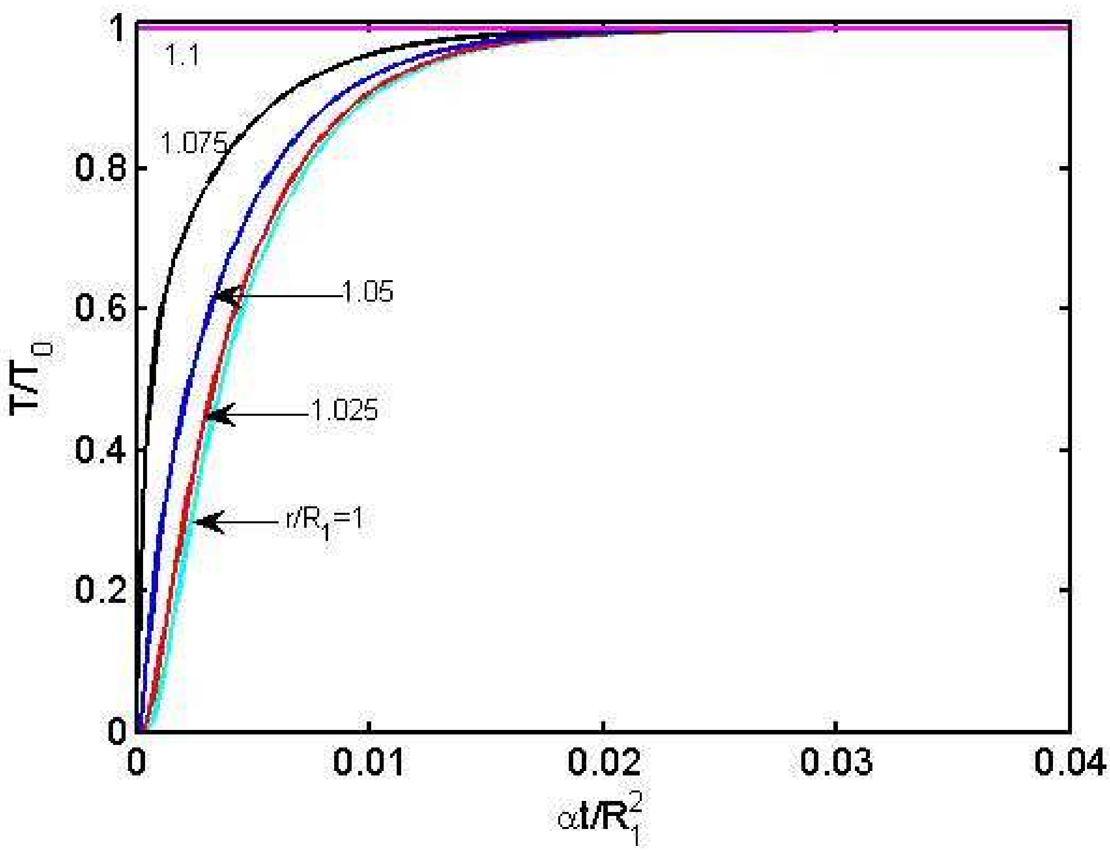

Fig. 2. Dimensionless transient temperatures versus dimensionless time for the R21B01T0 case with $\tilde{R} = 1.1$

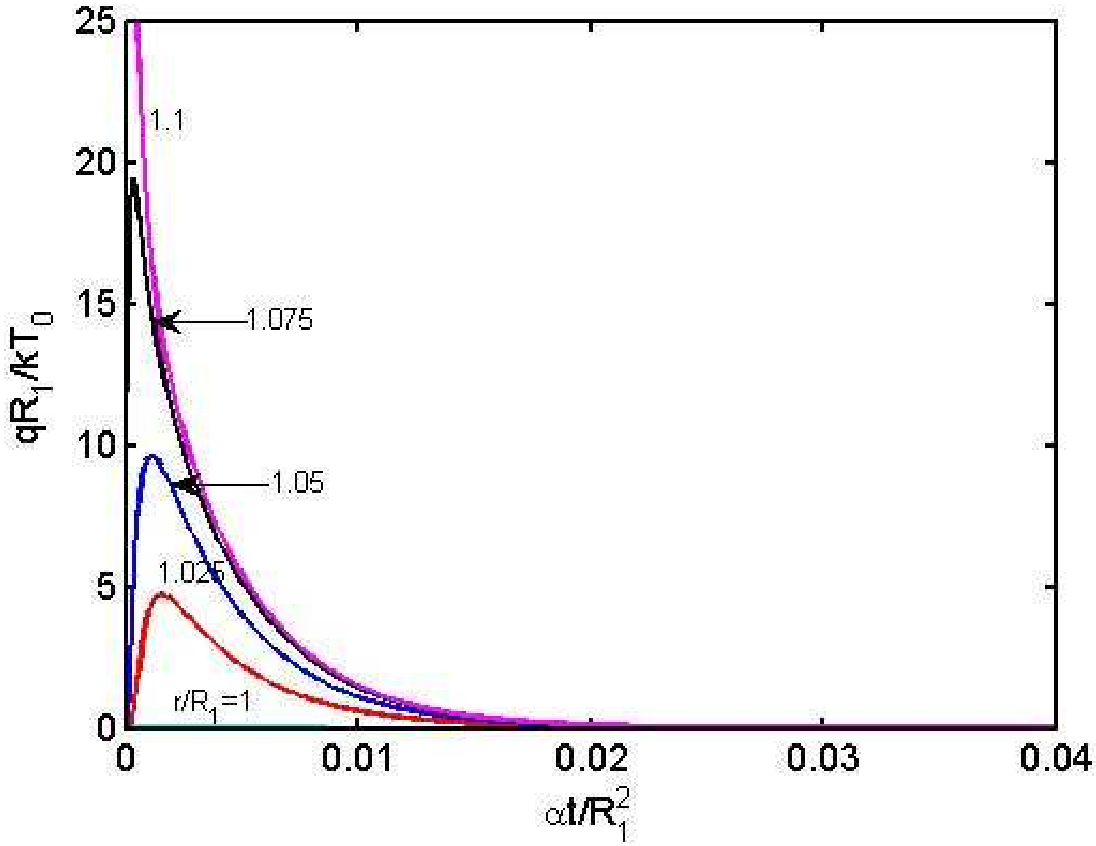

Fig. 3. Dimensionless transient heat fluxes versus dimensionless time for the R21B01T0 case with $\tilde{R} = 1.1$

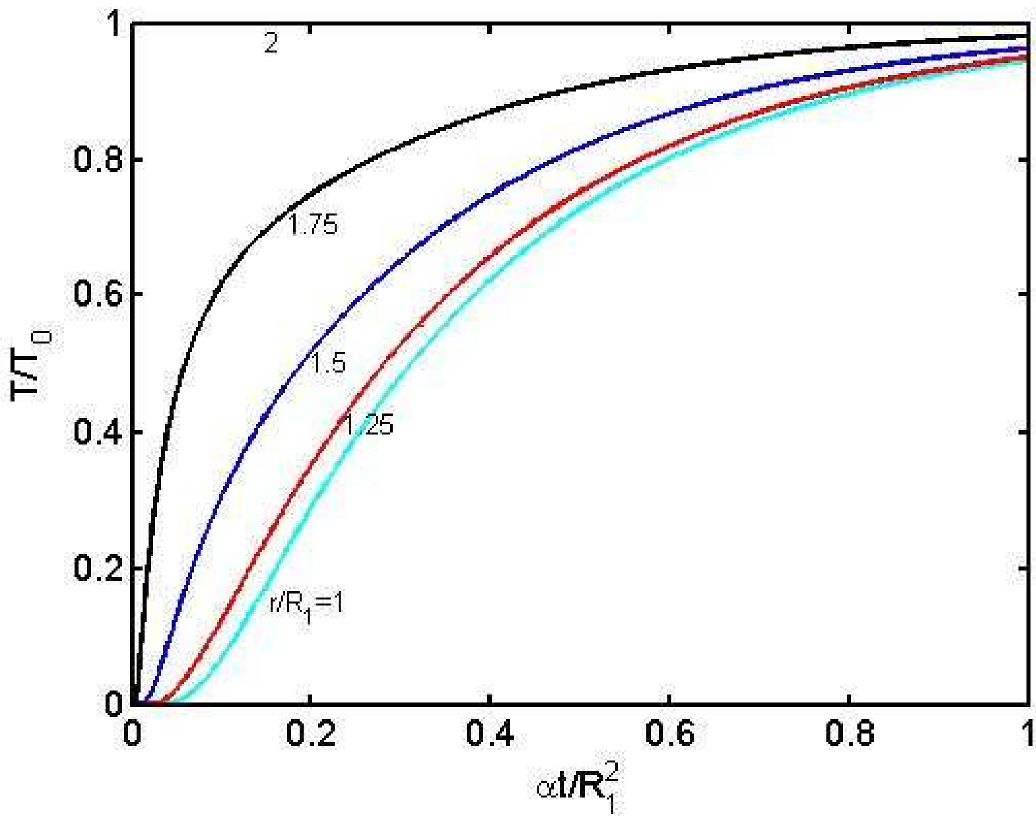

Fig. 4. Dimensionless transient temperatures versus dimensionless time for the R21B01T0 case with $\tilde{R}=2$.

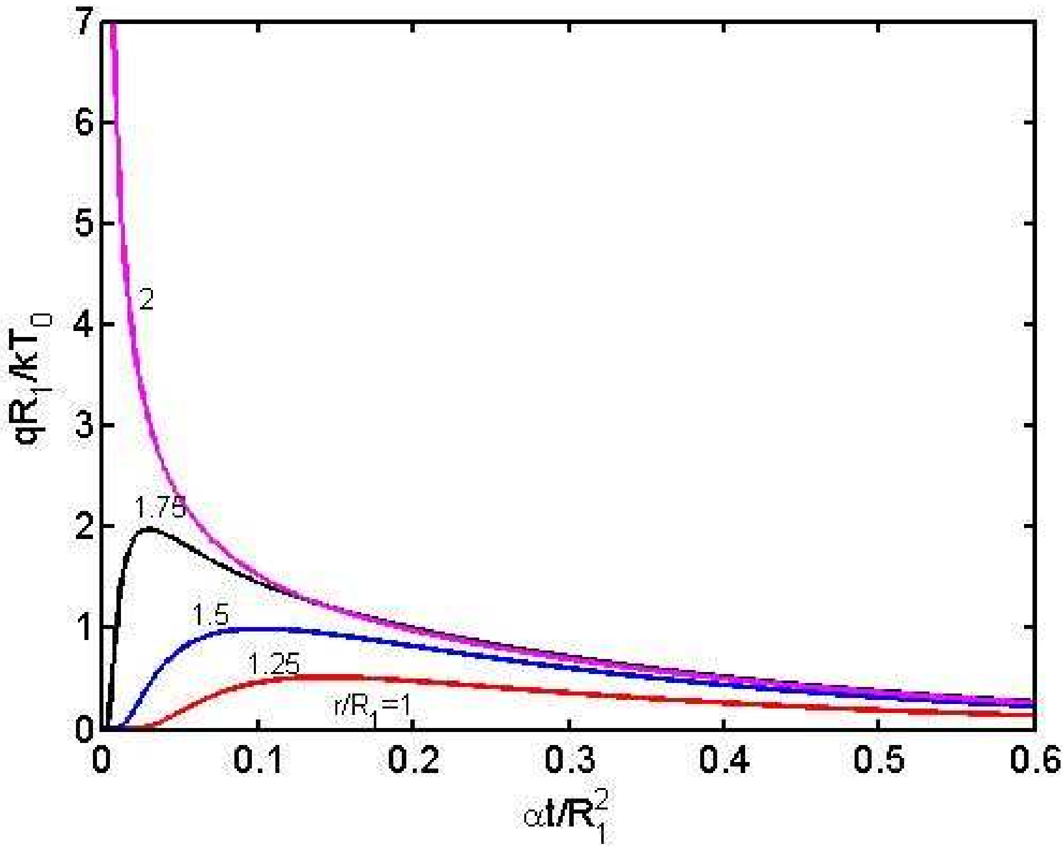

Fig. 5. Dimensionless transient heat fluxes versus dimensionless time for the R21B01T0 case with $\tilde{R} = 2$.