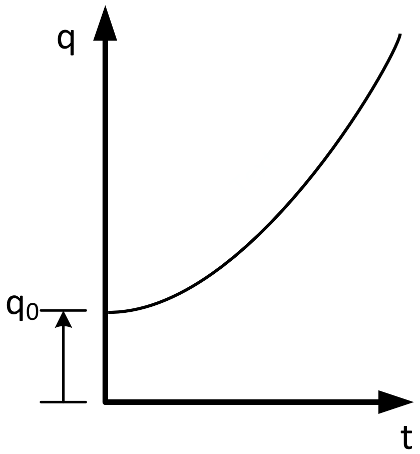
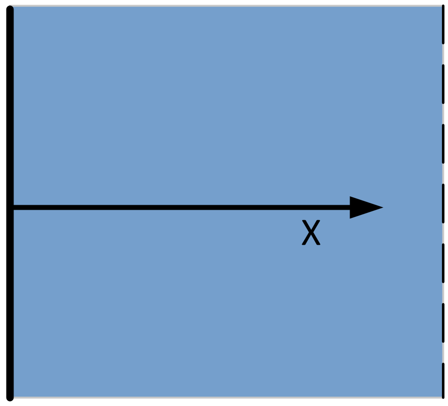
ExACT – Exact Analytical Conduction Toolbox

Temporal Exponential Ambient Temperature Problem, X20B4T0
James V. Beck and Jordan P. Krahn, Oct. 7, 2014

Consider a slab initially at T = 0. A heat flux with a positive exponential variation is applied at x = 0.





It is mathematically described by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad 0 < x < \infty, \quad t > 0$$

$$-k\frac{\partial T}{\partial x}(0,t) = q_0 e^{\gamma^2 \alpha t/L^2}, \quad T(\infty,t) \text{ is bounded} \qquad (1)$$

$$T(x,0) = 0$$

(The notation for this problem is X2B4T0.) For the negative exponential, see Appendix A by Don Amos. Solving this problem using Green's functions gives

$$T(x,t) = \frac{\alpha}{k} q_0 \int_{\tau=0}^{t} e^{\gamma^2 \frac{\alpha\tau}{L^2}} G_{X20}(x,0,t-\tau)d\tau = \frac{\alpha}{k} q_0 \int_{\tau=0}^{t} e^{\gamma^2 \frac{\alpha\tau}{L^2}} \frac{e^{-\frac{x^2}{4\alpha(t-\tau)}}}{\sqrt{\pi\alpha(t-\tau)}} d\tau$$

$$= \frac{\alpha}{k} q_0 e^{\gamma^2 \frac{\alpha t}{L^2}} \int_{\tau=0}^{t} e^{-\gamma^2 \frac{\alpha(t-\tau)}{L^2}} \frac{e^{-\frac{x^2}{4\alpha(t-\tau)}}}{\sqrt{\pi\alpha(t-\tau)}} d\tau \qquad (2)$$

Let us use the cotime defined by

$$u = t - \tau \qquad (3)$$

and rewrite eq. (2) as

$$T(x,t)=\frac{\alpha}{k}q_0 e^{\gamma^2\frac{\alpha t}{L^2}}\int_{u=0}^{t} e^{-\gamma^2\frac{\alpha u}{L^2}}\frac{e^{-\frac{x^2}{4\alpha u}}}{\sqrt{\pi\alpha u}}du=\frac{\alpha}{k}q_0 e^{\gamma^2\frac{\alpha t}{L^2}}\int_{u=0}^{t}\frac{e^{-\gamma^2\frac{\alpha u}{L^2}-\frac{x^2}{4\alpha u}}}{\sqrt{\pi\alpha u}}du$$
$$=\frac{\alpha}{k}q_0 e^{\gamma^2\frac{\alpha t}{L^2}}\int_{u=0}^{t}\frac{e^{-\gamma^2\frac{\alpha u}{L^2}-\frac{x^2}{4\alpha u}}}{\sqrt{\pi\alpha u}}du=\frac{\alpha}{k\sqrt{\pi\alpha}}q_0 e^{\gamma^2\frac{\alpha t}{L^2}}\int_{u=0}^{t}\frac{e^{-\bar{a}^2u-\frac{\bar{b}^2}{u}}}{\sqrt{u}}du \quad (4)$$

where

$$\bar{a}^2=\frac{\alpha\gamma^2}{L^2},\ \bar{b}^2=\frac{x^2}{4\alpha} \quad (5)$$

Now from Eq. (12) on page 514 of "Heat Conduction Using Green's Functions." 2nd Ed.,

$$\int_{u=0}^{t}\frac{e^{-\bar{a}^2u-\frac{\bar{b}^2}{u}}}{\sqrt{u}}du=\frac{\sqrt{\pi}}{2\bar{a}}\left[e^{2\bar{a}\bar{b}}\operatorname{erf}\left(\bar{a}\sqrt{u}+\frac{\bar{b}}{\sqrt{u}}\right)+e^{-2\bar{a}\bar{b}}\operatorname{erf}\left(\bar{a}\sqrt{u}-\frac{\bar{b}}{\sqrt{u}}\right)\right]_0^t$$
$$=\frac{\sqrt{\pi}}{2\bar{a}}\left[-e^{2\bar{a}\bar{b}}\operatorname{erfc}\left(\bar{a}\sqrt{t}+\frac{\bar{b}}{\sqrt{t}}\right)+e^{-2\bar{a}\bar{b}}\operatorname{erfc}\left(\frac{\bar{b}}{\sqrt{t}}-\bar{a}\sqrt{t}\right)\right] \quad (6)$$

At x = 0, $\bar{b}=0$ and then Eq. (6) gives

$$\int_{u=0}^{t}\frac{e^{-\bar{a}^2u}}{\sqrt{u}}du=\frac{\sqrt{\pi}}{2\bar{a}}\left[-\operatorname{erfc}\left(\bar{a}\sqrt{t}\right)+\operatorname{erfc}\left(-\bar{a}\sqrt{t}\right)\right]$$
$$=\frac{\sqrt{\pi}}{2\bar{a}}\left[-\operatorname{erfc}\left(\bar{a}\sqrt{t}\right)+2-\operatorname{erfc}\left(\bar{a}\sqrt{t}\right)\right]=\frac{\sqrt{\pi}}{\bar{a}}\operatorname{erf}\left(\bar{a}\sqrt{t}\right) \quad (7)^*$$

which is consistent with the result obtained from the same reference, page 513, # 5.

The complete solution of the X20B4T0 problem then is

$$T_{\text{X20B4T0}}(x,t)=\frac{\alpha}{k\sqrt{\pi\alpha}}q_0 e^{\gamma^2\frac{\alpha t}{L^2}}\frac{\sqrt{\pi}L}{2\gamma\sqrt{\alpha}}\left[\begin{matrix}-e^{\frac{x}{L}\gamma}\operatorname{erfc}\left(\dfrac{\gamma\sqrt{\alpha t}}{L}+\dfrac{x}{\sqrt{4\alpha t}}\right)\\ +e^{-\frac{x}{L}\gamma}\operatorname{erfc}\left(\dfrac{x}{\sqrt{4\alpha t}}-\dfrac{\gamma\sqrt{\alpha t}}{L}\right)\end{matrix}\right] \quad (8a)$$

In dimensionless terms Eq. (8a) becomes

$$\tilde{T}_{\text{X20B4T0}}(\tilde{x},\tilde{t})=\frac{1}{2\gamma}e^{\gamma^2\tilde{t}}\left[-e^{\tilde{x}\gamma}\operatorname{erfc}\left(\gamma\sqrt{\tilde{t}}+\frac{\tilde{x}}{\sqrt{4\tilde{t}}}\right)+e^{-\tilde{x}\gamma}\operatorname{erfc}\left(\frac{\tilde{x}}{\sqrt{4\tilde{t}}}-\gamma\sqrt{\tilde{t}}\right)\right] \quad (8b)$$

where

$$\tilde{T}_{\text{X20B4T0}}(\tilde{x},\tilde{t})=\frac{T_{\text{X20B4T0}}(x,t)}{\frac{q_0L}{k}},\ \tilde{x}=\frac{x}{L},\ \tilde{t}=\frac{\alpha t}{L^2} \quad (8c)$$

Notice that the orders of the arguments in Eq. (8b) are different in the two complementary error functions. One relatively simple case is for the heated surface, $\tilde{x}=0.$ Then Eq. (8b) reduces to

$$\tilde{T}_{\text{X20B4T0}}(0,\tilde{t}) = \frac{1}{\gamma} e^{\gamma^2 \tilde{t}} \text{erf}\left(\gamma\sqrt{\tilde{t}}\right) \tag{9a}$$

One concept in intrinsic verification is to check if a more complicated solution reduces to a simpler one. One such case is checking if Eq. (9a) reduces to the case of X20B1T0 when $\gamma \to 0$. Use Eq. (E-5b) in "Heat Conduction Using Green's Functions," page 496, 2nd Edition to get

$$\lim_{\gamma \to 0} \tilde{T}_{\text{X20B4T0}}(0,\tilde{t}) = \lim_{\gamma \to 0} \frac{1}{\gamma} e^{\gamma^2 \tilde{t}} \frac{2}{\sqrt{\pi}} e^{-\gamma^2 \tilde{t}} \gamma\sqrt{\tilde{t}} = 2\sqrt{\frac{\tilde{t}}{\pi}} \tag{9b}$$

This is the correct result for the surface temperature for the X20B1T0 case. This is a demonstration of intrinsic verification for this case. A Matlab program has been written for Eq. (8b) and some results are given in Tables 1 to 4. Table 1 is for $\gamma = 10^{-6}$ and the numerical values are close to those for $\gamma = 0$ which is again the same X20B1T0 problem but now results are demonstrated for $x > 0$; again intrinsic verification is demonstrated.

Note that Eq. (8b) also can be used for the X21B40T0, X22B40T0, X23B40T0, X24B40T0 and X25B40T0 cases for the small times given by

$$0 < \frac{\alpha t}{L^2} < \frac{(2-\tilde{x})^2}{10A}, \; \tilde{x} = x/L \tag{10}$$

The results should be accurate to about A decimal places.

MATLAB Codes

The function X20B4T0 calculates dimensionless temperature given dimensionless position, dimensionless temperature, and the parameter gamma

```
%X20B4T0.m James V. Beck and Jordan P. Krahn 9/24/2014
%Inputs:
%xd - dimensionless position. May be a vector.
%td - dimensionless time. May be a vector.
%gam - dimensionless conductivity. Must be a scalar.
%Output:
%Td - dimensionless temperature.
function [Td] = X20B4T0(xd, td, gam)

Td = zeros(length(td),length(xd));
for ix=1:length(xd);
    for it=1:length(td)
        Td(it,ix)=1/2/gam*exp(td(it)*gam^2).*(-exp(xd(ix)*gam)...
        .*erfc(gam*sqrt(td(it))+xd(ix)./sqrt(4*td(it)))+exp(-xd(ix)...
        *gam).*erfc(-gam*sqrt(td(it))+xd(ix)./sqrt(4*td(it))));
    end
end
```

The following script is used to create Table 1.

```
td=[.01:.01:.1   .15:.05:.5 .6:.1:.9  1:1:4];
xd=[0 .5 1];
gam=1.0e-6;
[Td] = X20B4T0(xd, td, gam);
BB = [td' Td]
sprintf('      time         T0             Tp5            T1')
fprintf(' %10.3f   %12.5f  %12.5f  %12.5f  \n',BB');
```

The following script is used to create Figure 1.

```
clear all
xd=[0,0.5,1]; td=0:.001:1;
[Td] = X20B4T0(xd, td,1e-6);
x = plot(td,Td,'Linewidth',2);
xlabel('Dimensionless Time');
ylabel('Dimensionless Temperature');
title('Figure 1')
l= legend('$\tilde{x}$ = 0.0','$\tilde{x}$ = 0.5','$\tilde{x}$ = 1.0');
set(l,'Interpreter','Latex');
```

Table 1. Dimensionless temperatures as a function of time at various positions with $\gamma = 10^{-6}$.

$\tilde{t}$	$\tilde{T}(0,\tilde{t})$	$\tilde{T}(0.5,\tilde{t})$	$\tilde{T}(1,\tilde{t})$
0.010	0.11284	0.00001	0.00000
0.020	0.15958	0.00080	0.00000
0.030	0.19544	0.00372	0.00000
0.040	0.22568	0.00875	0.00003
0.050	0.25231	0.01537	0.00013
0.060	0.27640	0.02307	0.00039
0.070	0.29854	0.03152	0.00087
0.080	0.31915	0.04047	0.00160
0.090	0.33851	0.04974	0.00263
0.100	0.35682	0.05922	0.00394
0.150	0.43702	0.10745	0.01465
0.200	0.50463	0.15459	0.03073
0.250	0.56419	0.19964	0.05025
0.300	0.61804	0.24251	0.07189
0.350	0.66756	0.28334	0.09480
0.400	0.71365	0.32234	0.11844
0.450	0.75694	0.35970	0.14246
0.500	0.79788	0.39559	0.16663
0.600	0.87404	0.46354	0.21489
0.700	0.94407	0.52713	0.26251
0.800	1.00925	0.58709	0.30919
0.900	1.07047	0.64396	0.35479
1.000	1.12838	0.69818	0.39928
2.000	1.59577	1.14538	0.79119
3.000	1.95441	1.49499	1.11505
4.000	2.25676	1.79193	1.39635

Table 2. Dimensionless temperatures as a function of time at various positions with $\gamma = 0.1$.

$\tilde{t}$	$\tilde{T}(0,\tilde{t})$	$\tilde{T}(0.5,\tilde{t})$	$\tilde{T}(1,\tilde{t})$
0.010	0.11285	0.00001	0.00000
0.020	0.15960	0.00080	0.00000
0.030	0.19548	0.00372	0.00000
0.040	0.22574	0.00876	0.00003
0.050	0.25240	0.01537	0.00013
0.060	0.27651	0.02308	0.00039
0.070	0.29868	0.03153	0.00087
0.080	0.31932	0.04048	0.00160
0.090	0.33872	0.04976	0.00263
0.100	0.35706	0.05924	0.00394
0.150	0.43746	0.10751	0.01466
0.200	0.50530	0.15472	0.03075
0.250	0.56513	0.19986	0.05029
0.300	0.61928	0.24283	0.07196
0.350	0.66912	0.28380	0.09491
0.400	0.71556	0.32295	0.11860
0.450	0.75921	0.36048	0.14268
0.500	0.80055	0.39656	0.16694
0.600	0.87754	0.46494	0.21539
0.700	0.94849	0.52903	0.26325
0.800	1.01465	0.58955	0.31021
0.900	1.07692	0.64704	0.35615
1.000	1.13593	0.70193	0.40102
2.000	1.61722	1.15855	0.79899
3.000	1.99397	1.52161	1.13255
4.000	2.31791	1.83537	1.42668
5.000	2.60894	2.11783	1.69420

Table 3. Dimensionless temperatures as a function of time at various positions with $\gamma = 1$

$\tilde{t}$	$\tilde{T}(0,\tilde{t})$	$\tilde{T}(0.5,\tilde{t})$	$\tilde{T}(1,\tilde{t})$
0.010	0.11359	0.00001	0.00000
0.020	0.16172	0.00080	0.00000
0.030	0.19940	0.00375	0.00000
0.040	0.23179	0.00884	0.00003
0.050	0.26089	0.01557	0.00014
0.060	0.28772	0.02347	0.00040
0.070	0.31287	0.03220	0.00088
0.080	0.33673	0.04152	0.00163
0.090	0.35957	0.05125	0.00267
0.100	0.38159	0.06129	0.00402
0.150	0.48346	0.11389	0.01518
0.200	0.57761	0.16808	0.03243
0.250	0.66834	0.22290	0.05410
0.300	0.75784	0.27829	0.07905
0.350	0.84749	0.33444	0.10659
0.400	0.93822	0.39159	0.13629
0.450	1.03072	0.44998	0.16791
0.500	1.12556	0.50986	0.20131
0.600	1.32409	0.63498	0.27324
0.700	1.53705	0.76868	0.35207
0.800	1.76729	0.91261	0.43819
0.900	2.01758	1.06843	0.53225
1.000	2.29070	1.23784	0.63502
2.000	7.05285	4.15323	2.41196
3.000	19.79820	11.89991	7.12026
4.000	54.34275	32.86337	19.84314
5.000	148.18083	89.78729	54.37560

Table 4. Dimensionless temperatures as a function of time at various positions with $\gamma = 10$.

$\tilde{t}$	$\tilde{T}(0,\tilde{t})$	$\tilde{T}(0.5,\tilde{t})$	$\tilde{T}(1,\tilde{t})$
0.010	0.22907	0.00002	0.00000
0.020	0.70529	0.00116	0.00000
0.030	1.97982	0.00785	0.00000
0.040	5.43428	0.02973	0.00005
0.050	14.81808	0.09199	0.00030
0.060	40.32142	0.26318	0.00122
0.070	109.64327	0.72983	0.00410
0.080	298.07692	1.99920	0.01237
0.090	810.29049	5.45029	0.03533
0.100	2202.62952	14.83170	0.09825

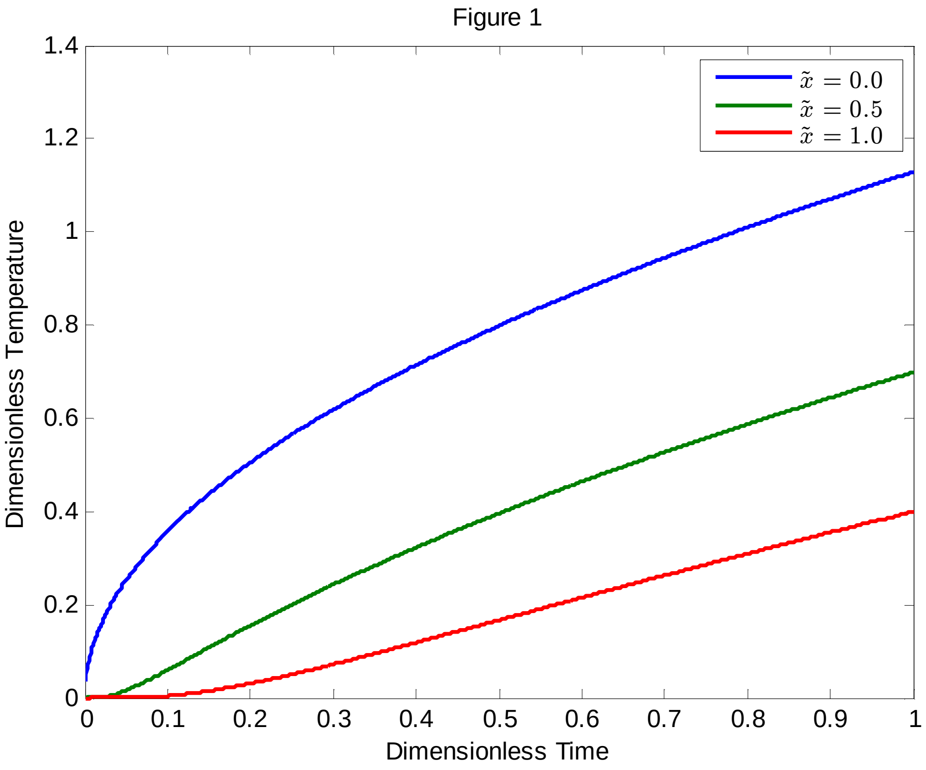


Figure 1. Temperature as a function of time for various positions with $\gamma = 10^{-6}$.

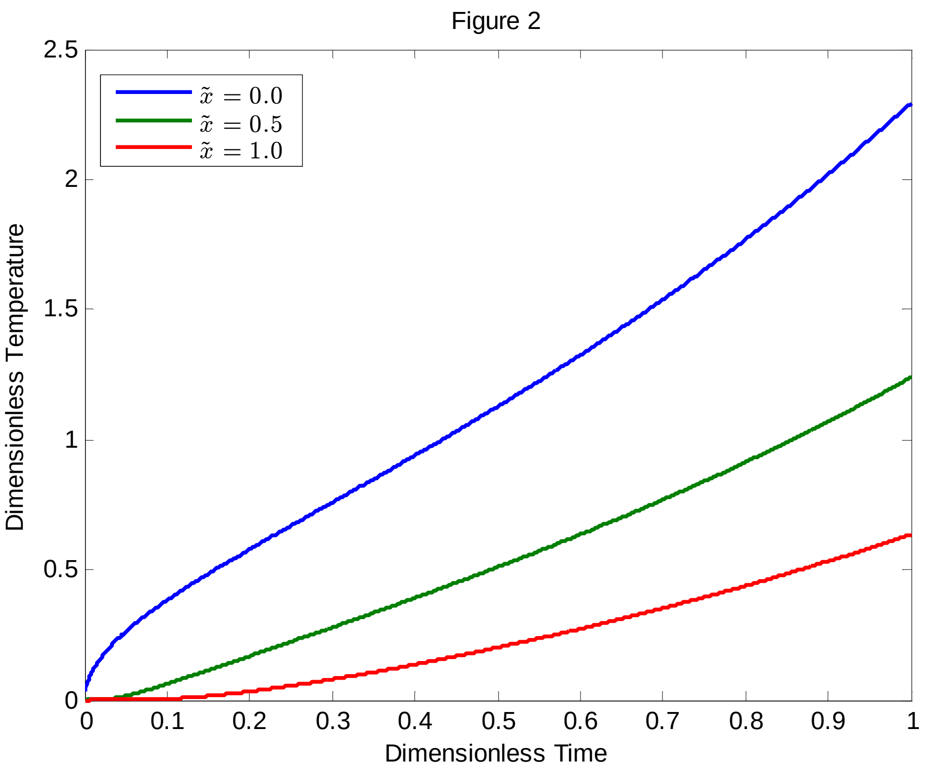


Figure 2. Temperature as a function of time for various positions with $\gamma = 1$.

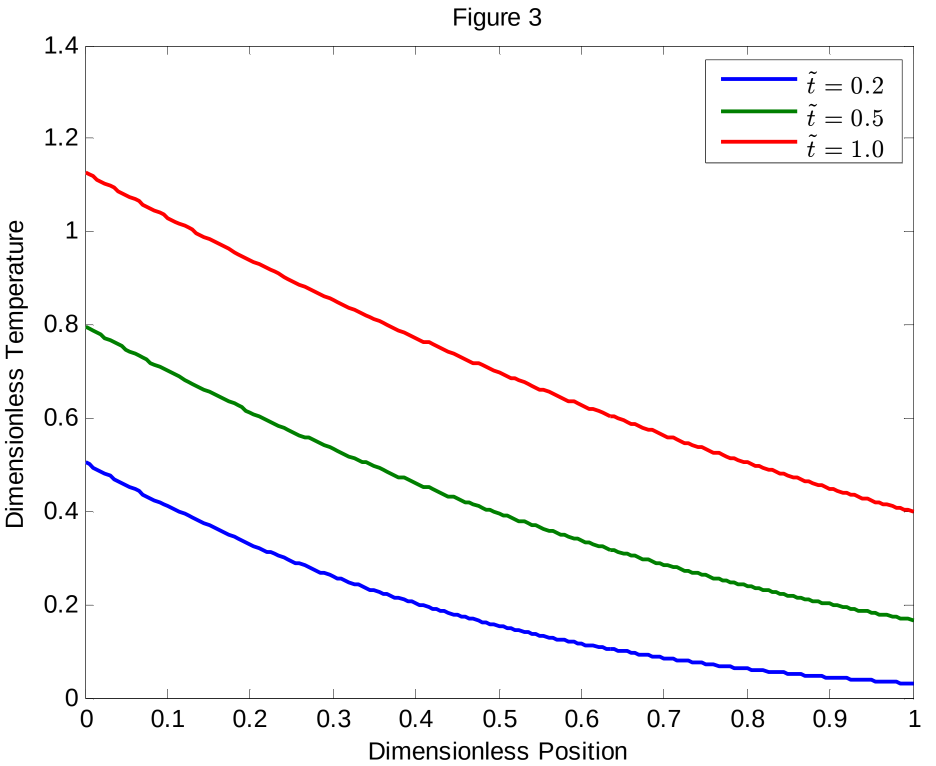


Figure 3. Temperature as a function of position for various times with $\gamma = 10^{-6}$.

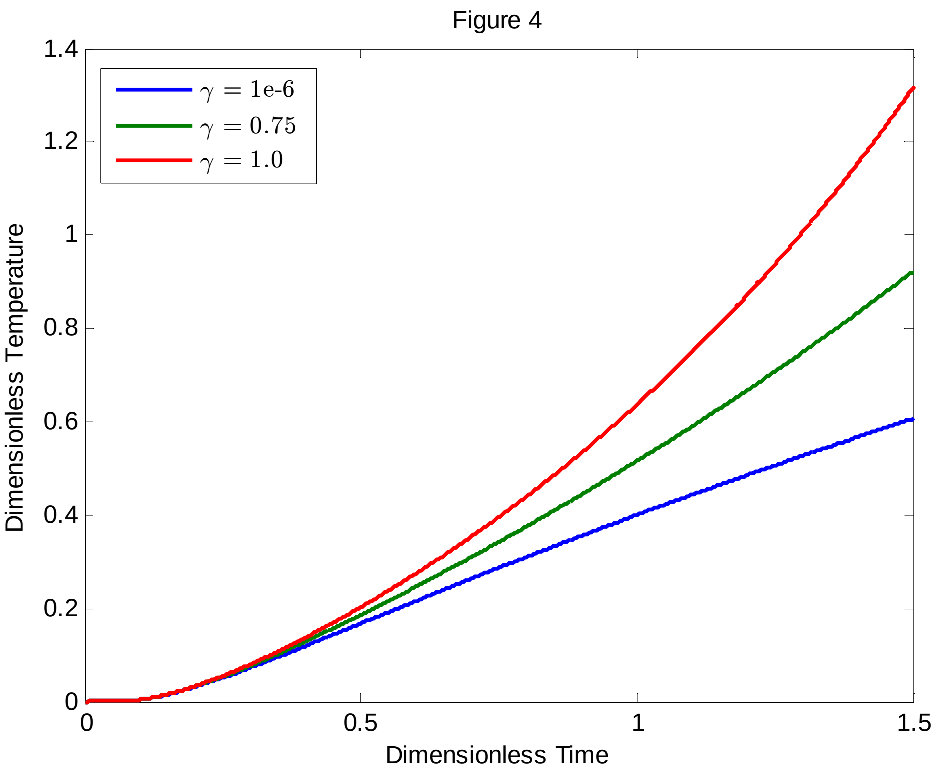


Figure 4. Temperature as a function of time for several values of γ with $\tilde{x} = 1.0$.

Appendix A. Beck's Complex Error Function Integral

Donald E. Amos, 1/31/2014

(1)
$$\int_{u=0}^{t} \frac{e^{-\bar{a}^2 u - \frac{\bar{b}^2}{u}}}{\sqrt{u}} du = \frac{\sqrt{\pi}}{2\bar{a}} \left[e^{2\bar{a}\bar{b}} \operatorname{erf}\left(\bar{a}\sqrt{u} + \frac{\bar{b}}{\sqrt{u}} \right) + e^{-2\bar{a}\bar{b}} \operatorname{erf}\left(\bar{a}\sqrt{u} - \frac{\bar{b}}{\sqrt{u}} \right) \right]_0^t$$
$$= \frac{\sqrt{\pi}}{2\bar{a}} \left[-e^{2\bar{a}\bar{b}} \operatorname{erfc}\left(\bar{a}\sqrt{t} + \frac{\bar{b}}{\sqrt{t}} \right) + e^{-2\bar{a}\bar{b}} \operatorname{erfc}\left(\frac{\bar{b}}{\sqrt{t}} - \bar{a}\sqrt{t} \right) \right]$$

Replace $\bar{a}$ *with ia and* $\bar{b}$ *with b*

(2)
$$\int_{u=0}^{t} \frac{e^{a^2 u - \frac{b^2}{u}}}{\sqrt{u}} du = \frac{\sqrt{\pi}}{2ia} \left[e^{2iab} \operatorname{erf}\left(ia\sqrt{u} + \frac{b}{\sqrt{u}} \right) + e^{-2iab} \operatorname{erf}\left(ia\sqrt{u} - \frac{b}{\sqrt{u}} \right) \right]_0^t$$
$$= \frac{\sqrt{\pi}}{2ia} \left[-e^{2iab} \operatorname{erfc}\left(ia\sqrt{t} + \frac{b}{\sqrt{t}} \right) + e^{-2iab} \operatorname{erfc}\left(\frac{b}{\sqrt{t}} - ia\sqrt{t} \right) \right]$$

Now each argument is large in magnitude for both small t and large t. Therefore we can use the asymptotic expansion for large (complex) argument for each of the error functions, (A&S, p300)

(3)
$$erfc(z) \sim \frac{e^{-z^2}}{z\sqrt{\pi}} \sum_{m=0}^{M} \frac{(-1)^m (1/2)_m}{z^{2m}} = \frac{e^{-z^2}}{z\sqrt{\pi}} - \frac{(1/2)e^{-z^2}}{z^3\sqrt{\pi}} + \frac{(1/2)(3/2)e^{-z^2}}{z^5\sqrt{\pi}} - \ldots \quad |\arg(z)| < 3\pi/4$$
$$(1/2)_0 = 1 \quad , \quad (1/2)_m = (1/2)(3/2)\ldots(m-1/2) \ , \ \text{m=1,2,...}$$

The first term (m=0) is

$$\int_{u=0}^{t} \frac{e^{a^2 u - \frac{b^2}{u}}}{\sqrt{u}} du = \frac{\sqrt{\pi}}{2ia} \left[-e^{2iab} \operatorname{erfc}\left(ia\sqrt{t} + \frac{b}{\sqrt{t}} \right) + e^{-2iab} \operatorname{erfc}\left(\frac{b}{\sqrt{t}} - ia\sqrt{t} \right) \right]$$

(4)

$$: -\frac{\sqrt{\pi}}{2ia} \left[\frac{e^{2iab} \mathrm{e}^{-(ia\sqrt{t} + \frac{b}{\sqrt{t}})^2}}{(ia\sqrt{t} + \frac{b}{\sqrt{t}})\sqrt{\pi}} - \frac{e^{-2iab} \mathrm{e}^{-(\frac{b}{\sqrt{t}} - ia\sqrt{t})^2}}{(\frac{b}{\sqrt{t}} - ia\sqrt{t})\sqrt{\pi}} \right]$$

In this expression the second term is the complex conjugate of the first term. Therefore the difference is 2i*(Imaginary part of the first term). To get a ratio of complex numbers into the standard form of a+bi, multiply the denominator by its conjugate:

(5)

$$: -\frac{1}{2ia}2i\,\mathrm{Im}\left[\frac{e^{2iab}\mathrm{e}^{-(ia\sqrt{t}+b/\sqrt{t})^2}}{(b/\sqrt{t}+ia\sqrt{t})}\right] = -\frac{1}{a}\mathrm{Im}\frac{e^{a^2t-b^2/t}(b/\sqrt{t}-ia\sqrt{t})}{(b/\sqrt{t}+ia\sqrt{t})(b/\sqrt{t}-ia\sqrt{t})}$$

$$= -\frac{1}{a}\frac{-a\sqrt{t}e^{a^2t-b^2/t}}{(b^2/t+a^2t)} = \frac{\sqrt{t}e^{a^2t-b^2/t}}{(b^2/t+a^2t)}.$$

Then the first term (m=0) is,

(6)

$$\int_{u=0}^{t}\frac{e^{a^2u-\frac{b^2}{u}}}{\sqrt{u}}du : \frac{\sqrt{t}e^{a^2t-b^2/t}}{(b^2/t+a^2t)}$$

for both small and large t. Since the real and imaginary parts of the complex numbers are in the right half plane (real part >0), the error function arguments (in magnitude) are less than $\pi/2 < 3\pi/4$. The succeeding terms are easy since the numerator is a real number. To get the next term multiply the numerator and denominator by the conjugate $(b/\sqrt{t}-ia\sqrt{t})^3$ to get a real number in the denominator and the imaginary part in the numerator for the standard form of a complex number. Then we expand the binomial expression $(b/\sqrt{t}-ia\sqrt{t})^3$ into its polynomial form with the binomial expansion. Then for m=1,

(7)

$$-\frac{(1/2)e^{-z^2}}{z^3\sqrt{\pi}} : \quad \frac{(1/2)}{2ia}2i\,\mathrm{Im}\left[\frac{e^{2iab}\mathrm{e}^{-(ia\sqrt{t}+b/\sqrt{t})^2}}{(b/\sqrt{t}+ia\sqrt{t})^3}\right] = \frac{1}{2a}\mathrm{Im}\frac{e^{a^2t-b^2/t}(b/\sqrt{t}-ia\sqrt{t})^3}{\left[(b/\sqrt{t}+ia\sqrt{t})(b/\sqrt{t}-ia\sqrt{t})\right]^3}$$

$$= \frac{1}{2a}\mathrm{Im}\frac{e^{a^2t-b^2/t}\left[(b/\sqrt{t})^3 - 3i(b/\sqrt{t})^2(a\sqrt{t}) - 3(b/\sqrt{t})(a\sqrt{t})^2 + i(a\sqrt{t})^3\right]}{(b^2/t+a^2t)^3}$$

$$= \frac{1}{2a}\frac{e^{a^2t-b^2/t}\left[-3(b/\sqrt{t})^2(a\sqrt{t}) + (a\sqrt{t})^3\right]}{(b^2/t+a^2t)^3} = -\frac{1}{2a}\frac{e^{a^2t-b^2/t}\left[3b^2/t - a^2t\right](a\sqrt{t})}{(b^2/t+a^2t)^3}$$

The dimension on the integral is $\sqrt{t}$ and both terms for m=0 and m=1 have dimension $\sqrt{t}$. For succeeding terms, the general expression is

$$\int_{u=0}^{t}\frac{e^{a^2u-\frac{b^2}{u}}}{\sqrt{u}}du = \sum_{m=0}^{M}(-1)^m(1/2)_m\left\{-\frac{1}{2ia}2i\,\mathrm{Im}\left[\frac{e^{2iab}\mathrm{e}^{-(ia\sqrt{t}+b/\sqrt{t})^2}}{(b/\sqrt{t}+ia\sqrt{t})^{2m+1}}\right]\right\}$$

Since the numerator is real, one has to pick out the imaginary part of $1/z^{2m+1}, \quad m=0,1,2,...,M$. We multiply the numerator and denominator by the conjugate and take the odd terms of the binomial expansion to get the imaginary part (C_k^n is a binomial coefficient)

(8)

$$\mathrm{Im}\frac{(b/\sqrt{t}-ia\sqrt{t})^{2m+1}}{(b/\sqrt{t}+ia\sqrt{t})^{2m+1}(b/\sqrt{t}-ia\sqrt{t})^{2m+1}}=\frac{\mathrm{Im}\sum_{k=0}^{2m+1} C_k^{2m+1}(b/\sqrt{t})^{2m+1-k}(-ia\sqrt{t})^k}{(b^2/t+a^2t)^{2m+1}}$$

$$=\frac{\frac{1}{i}\sum_{k=0}^{m} C_{2k+1}^{2m+1}(b/\sqrt{t})^{2m-2k}(-ia\sqrt{t})^{2k+1}}{(b^2/t+a^2t)^{2m+1}}=\frac{-\sum_{k=0}^{m}(-1)^k C_{2k+1}^{2m+1}(b/\sqrt{t})^{2m-2k}(a\sqrt{t})^{2k+1}}{(b^2/t+a^2t)^{2m+1}}$$

The final expansion for either large t or small t is

(9)

$$\int_0^t \frac{e^{a^2u-b^2/u}}{\sqrt{u}}du : \frac{e^{a^2t-b^2/t}}{a}\sum_{m=0}^{M}\frac{(-1)^m(1/2)_m}{(b^2/t+a^2t)^{2m+1}}\sum_{k=0}^{m}(-1)^k C_{2k+1}^{2m+1}(b/\sqrt{t})^{2m-2k}(a\sqrt{t})^{2k+1}.$$

The dimension checks out since 1/a has dimension $\sqrt{t}$ and all other variables are dimensionless.
Beck chooses the representation

(10) $$\int_0^t \frac{e^{a^2u-b^2/u}}{\sqrt{u}}du : \frac{\sqrt{t}e^{a^2t-b^2/t}}{\frac{b^2}{t}+a^2t}\sum_{m=0}^{M}\frac{(-1)^m(1/2)_m\left(\frac{b^2}{t}\right)^m}{\left(\frac{b^2}{t}+a^2t\right)^{2m}}\sum_{k=0}^{m}(-1)^k C_{2k+1}^{2m+1}\left(\frac{a}{b}\right)^{2k}$$

Check for a=0

$$\int_0^t \frac{e^{-b^2/u}}{\sqrt{u}}du \overset{u=t/v}{=} \int_1^\infty \frac{e^{-b^2v/t}}{\sqrt{t}}\sqrt{v}\frac{t}{v^2}dv = \sqrt{t}\int_1^\infty \frac{e^{-b^2v/t}}{v^{3/2}}dv=\sqrt{t}E_{3/2}(b/\sqrt{t})=2\sqrt{\pi t}\, ierfc(b/\sqrt{t})$$

The hand book (A&S, p300, n=1, (7.2.14)) gives

$$ierfc(z)=\frac{e^{-z^2}}{2\sqrt{\pi}z^2}\sum_{m=0}^{M}\frac{(-1)^m\Gamma(2m+2)}{\Gamma(m+1)2^{2m}z^{2m}} \quad ,$$

$$\frac{\Gamma(2m+2)}{\Gamma(m+1)}=\frac{2^{2m+1}}{\sqrt{\pi}}\frac{\Gamma(m+1)}{\Gamma(m+1)}\frac{\Gamma(m+3/2)}{\Gamma(3/2)}\Gamma(3/2)=\frac{2^{2m+1}}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2}(3/2)_m=2^{2m}(3/2)_m$$

$$ierfc(z)=\frac{e^{-z^2}}{2\sqrt{\pi}z^2}\sum_{m=0}^{M}\frac{(-1)^m(3/2)_m}{z^{2m}}$$

For $z = b/\sqrt{t}$

$$\int_0^t \frac{e^{-b^2/u}}{\sqrt{u}} du = 2\sqrt{\pi t} ierfc(b/\sqrt{t}) : \frac{\sqrt{t}e^{-b^2/t}}{(b^2/t)} \sum_{m=0}^{M} \frac{(-1)^m (3/2)_m}{(b^2/t)^m}$$

Now, formula (10) with a=0 gives(k=0 is the only term contributing to the k sum)

$$\int_0^t \frac{e^{-b^2/u}}{\sqrt{u}} du : \frac{\sqrt{t}e^{-b^2/t}}{\frac{b^2}{t}+a^2 t} \sum_{m=0}^{M} \frac{(-1)^m (1/2)_m \left(\frac{b^2}{t}\right)^m}{\left(\frac{b^2}{t}\right)^{2m}} C_1^{2m+1} \quad , \quad C_1^{2m+1} = 2m+1 = 2(m+1/2)$$

$$: \frac{\sqrt{t}e^{-b^2/t}}{b^2/t} \sum_{m=0}^{M} \frac{(-1)^m (1/2)_m 2(m+1/2)}{\left(b^2/t\right)^m}$$

$$(1/2)_m 2(m+1/2) = (1/2)(3/2)...(m-1/2)2(m+1/2) = (3/2)_m$$

$$: \frac{\sqrt{t}e^{-b^2/t}}{b^2/t} \sum_{m=0}^{M} \frac{(-1)^m (3/2)_m}{\left(b^2/t\right)^m}$$

The two formulas agree.

Beck:

Binomial coefficients, p. 43 Spanier and Oldham

$$C_{2k+1}^{2m+1} = \prod_{j=0}^{2k} \frac{2m+1-j}{2k+1-j}$$

$$k=0,\ C_1^{2m+1} = \prod_{j=0}^{0} \frac{2m+1-j}{1-j} = 2m+1$$

$$k=1,\ C_3^{2m+1} = \prod_{j=0}^{0} \frac{2m+1-j}{3-j} = \frac{(2m+1)2m(2m-1)}{3\cdot 2\cdot 1}$$

Pockhammer polynomial $(1/2)_n$, p. 153, S and Oldham

$$(1/2)_0 = 1,\ (1/2)_n = \frac{(2n)!}{4^n n!}$$