Exact Analytical Conduction Toolbox

X22B10T00, Thermal wave in a slab with an applied constant heat flux at *x*=0, insulated at *x*=*L*, zero initial temperature and its time derivative, having constant thermo-physical properties.

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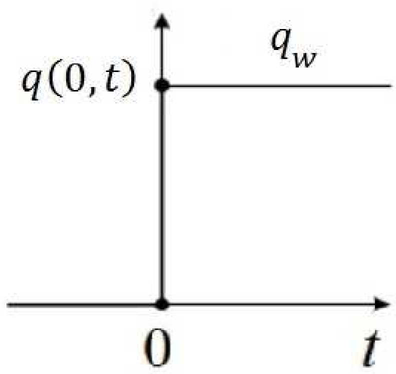

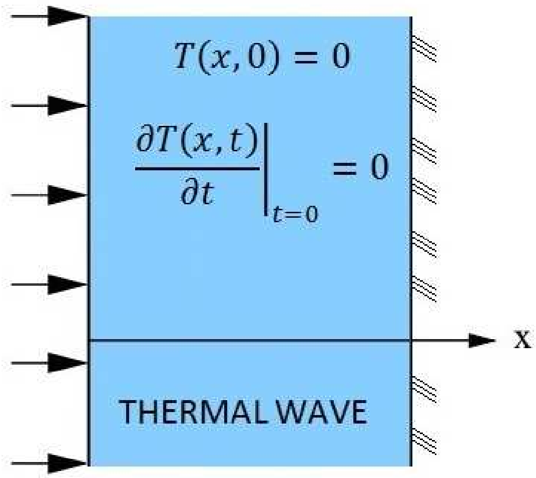

Dimensional mathematical statement for a **X22B10T00** problem

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau_q}{\alpha}\frac{\partial^2 T}{\partial t^2},\quad 0 < x < L \text{ and } t > 0 \qquad \text{(X22B10-1)}$$

where α is the thermal diffusivity and $\tau_q = \alpha / \sigma^2$ where σ is the thermal wave speed. The initial and boundary conditions for the reduced temperature $T(x,t)$ are

$$q(0,t) = q_w \qquad \text{(X22B10-2a)}$$

$$\partial T(x,t)/\partial x\big|_{t=L} = 0 \qquad \text{(X22B10-2b)}$$

$$T(x,0) = 0 \qquad \text{(X22B10-2c)}$$

$$\partial T(x,t)/\partial t\big|_{t=0} = 0 \qquad \text{(X22B10-2d)}$$

Dimensionless problem **X22B20T00**

$$\frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} = \frac{\partial \tilde{T}}{\partial \tilde{t}} + \tilde{\tau}_q \frac{\partial^2 \tilde{T}}{\partial \tilde{t}^2},\quad 0 < \tilde{x} < L \text{ and } \tilde{t} > 0 \qquad \text{(X22B10-3)}$$

$$\tilde{q}(0,\tilde{t}) = 1 \qquad \text{(X22B10-4a)}$$

$$\partial \tilde{T}(\tilde{x},\tilde{t})/\partial \tilde{x}\big|_{\tilde{x}=1} = 0 \qquad \text{(X22B10-4b)}$$

$$\tilde{T}(\tilde{x},0) = 0 \qquad \text{(X22B10-4c)}$$

$$\partial \tilde{T}(\tilde{x},\tilde{t}) / \partial \tilde{t}\Big|_{\tilde{t}=0} = 0 \qquad \text{(X22B10-4d)}$$

Dimensionless groups for the **X22B10T00** problem are

$$\tilde{T}(\tilde{x},\tilde{t}) = \frac{k\theta(\xi,\eta)}{q_0 L}, \ \tilde{q} = \frac{q(x,t)}{q_0}, \ \tilde{x} = \frac{x}{L}, \ \tilde{t} = \frac{\alpha t}{L^2}, \ \tilde{\tau}_q = \frac{\alpha \tau_q}{L^2} \qquad \text{(X22B10-5)}$$

When $\tau_q \neq 0$, Eq. (X12B10-3) can be written as

$$\frac{\partial^2 \tilde{T}}{\partial \xi^2} = \frac{\partial \tilde{T}}{\partial \eta} + \frac{\partial^2 \tilde{T}}{\partial \eta^2} \qquad \text{(X22B10-6)}$$

with $\xi = \tilde{x} / \sqrt{\tilde{\tau}_q}$ and $\eta = \tilde{t} / \tilde{\tau}_q$. This equation has a finite series solution, that is

$$\begin{aligned}\tilde{T}(\xi,\eta) &= \sum_{m=0}^{M} \sqrt{\tilde{\tau}_q}\,\{H_0[\phi_{1m},\eta] + H_0[\phi_{2m},\eta]\} \\ &+ \sqrt{\tilde{\tau}_q}\exp\left(-\frac{\eta}{2}\right)\sum_{m=0}^{M}\left\{I_0\left[\frac{1}{2}\sqrt{\eta^2-\phi_{1m}^2}\right]u[\tilde{t}-\phi_{1m}]\right. \\ &\left. + I_0\left[\frac{1}{2}\sqrt{\eta^2-\phi_{2m}^2}\right]u[\eta-\phi_{2m}]\right\}\end{aligned} \qquad \text{(X22B10-7a)}$$

where *u* is the unit step function, the number of terms within this series is $M = \tilde{t} / \sqrt{\tilde{\tau}_q} = \sigma t / L$ rounded to nearest integer while

$$\phi_{1m} = \frac{2m+\tilde{x}}{\sqrt{\tilde{\tau}_q}} = \frac{2m}{\sqrt{\tilde{\tau}_q}} + \xi, \ \phi_{2m} = \frac{2(m+1)-\tilde{x}}{\sqrt{\tilde{\tau}_q}} = \frac{2(m+1)}{\sqrt{\tilde{\tau}_q}} - \xi, \text{ and}$$

$$H_0(\phi,\eta) = \int_{\phi}^{\eta} e^{-\eta/2}\, I_0\left[\frac{1}{2}\sqrt{\eta^2-\phi^2}\right] d\eta, \text{ when } \eta > \phi \qquad \text{(X22B10-7b)}$$

wherein ϕ stands for ϕ_{1m} and ϕ_{2m}. When time is finite, the rounded parameter $M = \sigma t / L$ has a finite value and the series solution is exact. The local heat flux, using Cattaneo-Vernotte, equation

$$q(x,t) + \tau_q \frac{\partial q(x,t)}{\partial t} = -k\frac{\partial T(x,t)}{\partial t}$$

is

$$\begin{aligned}\tilde{q} = \sum_{m=0}^{M} &\left\{ e^{-\phi_{1m}(\xi)/2} u[\tilde{t} - \phi_{1m}(\xi)] + \frac{\phi_{1m}(\xi)}{2} H_1[\phi_{1m}(\xi),\eta] \right. \\ &\left. + e^{-\phi_{2m}(\xi)/2} u[\tilde{t} - \phi_{2m}(\xi)] + \frac{\phi_{2m}(\xi)}{2} H_1[\phi_{2m}(\xi),\eta] \right\}\end{aligned} \qquad \text{(X22B10-8)}$$

The function $H_\nu(\phi,\eta)$, appearing In Eq. (X22B10-7a) with ν=0 and in Eq. (X22B10-8) with ν=1, has a rapidly converging series solution defined, when $\eta > \phi$, as

$$H_\nu(\phi,\eta) = \int_{\eta=\phi}^{\eta} \frac{e^{-\eta/2} I_\nu\left[\frac{1}{2}\sqrt{\eta^2-\phi^2}\right]}{\left(\eta^2-\phi^2\right)^{\nu/2}} d\eta$$

(X22B10-8)

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+\nu-1}} \left(\frac{\phi}{2}\right)^n \frac{\Psi_{n+\nu}(\phi) - \Psi_{n+\nu}(\eta)}{2(n+\nu)-1}$$

with $\boldsymbol{\Psi}_{n+\nu}(\eta) = e^{-\eta}\left(\frac{I_{n+\nu}(\eta)+I_{n+\nu-1}(\eta)}{\eta^{n+\nu-1}}\right)$ while the dummy variable η is replaced by the upper limit $\eta = \tilde{t}/\tilde{\tau}_q$ and the lower limits ϕ that stands for $\phi_{1m}(\tilde{x})$ and $\phi_{2m}(\tilde{x})$. As an illustration; the numerical integration and a seven-term solution provided the same result, $H_0(1, 2)$ =0.5144221248465258. To show the convergence behavior, accurate digits of a two-term solution and a five-term solution are 0.5144221 and 0.5144221248, respectively. In the following numerical illustrations, the solutions of the thermal wave equation are compared to those from Fourier-type thermal conduction solutions.

The attached Mathematica program provided temperature values as they arrive to the surface at $x=L$ and then depart. The temperature values, plotted in Figure 1, show a unique jump in the temperature values as it returns from insulated $x=L$ surface. This represents the accumulation of the kinetic energy in a smaller space and the second law of thermodynamics does not apply to this case. A sample of acquired temperature values are in Table 1, also for $\tilde{\tau}_q = 1$. It is to be noted that the front arrives to $x=L$ location when $\tilde{\tau} = 1$**,** as can be detected from Figure 1. Then, as shown in in columns 5 and 6, temperature rapidly increases**.**

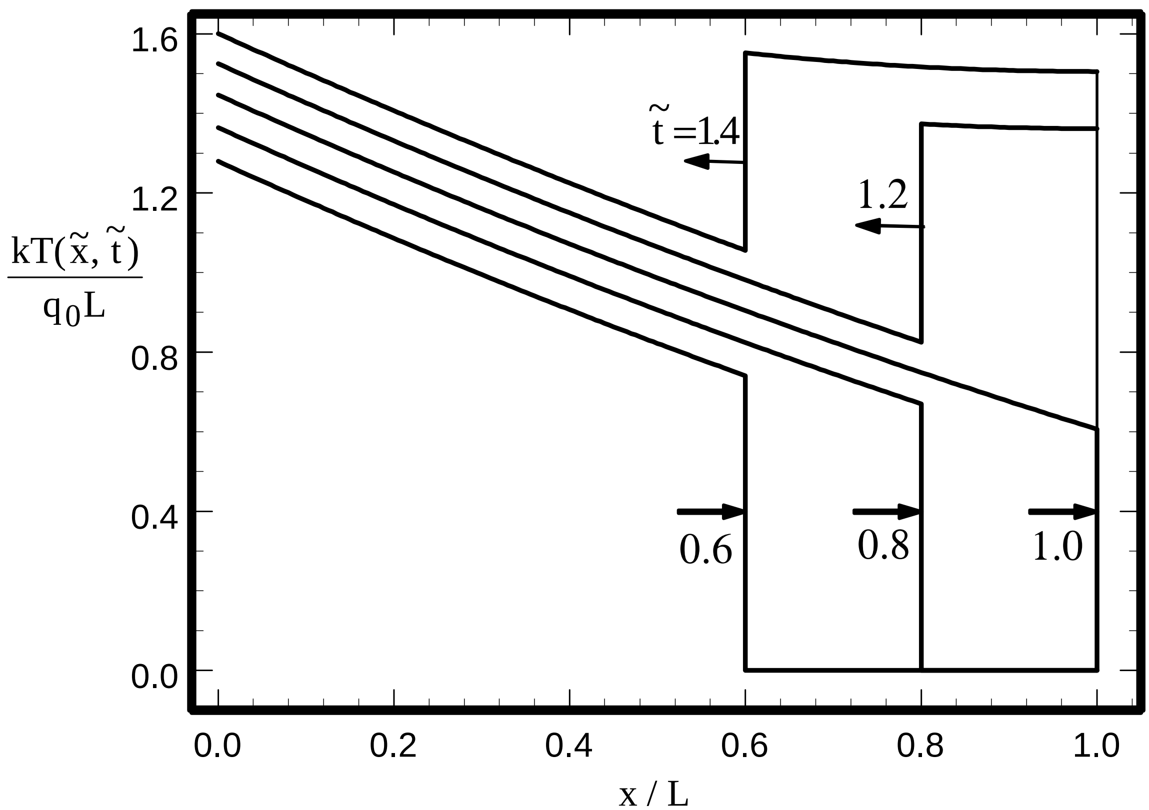

Figure 1. Temperature $\tilde{T}(\tilde{x}, \tilde{t})$ as a function of $\tilde{x} = x/L$ at specified time $\tilde{t}$ and the viewing of traveling front across the layer, when $\tilde{\tau}_q = 1$.

Table 1. Computed temperature $\tilde{T}(\tilde{x},\tilde{t})$ when $\tilde{\tau}_q = 1$.

$\tilde{x} = \frac{x}{L}$	$\tilde{t} = 0.6$	$\tilde{t} = 0.8$	$\tilde{t} = 1.0$	$\tilde{t} = 1.2$	$\tilde{t} = 1.4$
0.00	1.27956	1.36473	1.44650	1.52512	1.60087
0.05	1.22998	1.31515	1.39690	1.47551	1.55125
0.10	1.18126	1.26639	1.34811	1.42669	1.50240
0.15	1.13339	1.21846	1.30012	1.37865	1.45431
0.20	1.09104	1.17604	1.25763	1.33609	1.41169
0.25	1.04022	1.12509	1.20657	1.28493	1.36043
0.30	0.99490	1.07965	1.16100	1.23924	1.31463
0.35	0.95044	1.03503	1.11623	1.19433	1.2696
0.40	0.90683	0.99123	1.07227	1.15021	1.22533
0.45	0.86406	0.94826	1.0291	1.10686	1.18181
0.50	0.82214	0.90610	0.98673	1.0643	1.13906
0.55	0.78106	0.86477	0.94516	1.0225	1.09706
0.60	0	0.82426	0.90439	0.98149	1.05581
0.65	0	0.78456	0.86441	0.94125	1.54144
0.70	0	0.74567	0.82522	0.90177	1.53193
0.75	0	0.70759	0.78682	0.86307	1.52388
0.80	0	0	0.7492	0.82513	1.51729
0.85	0	0	0.71237	1.36869	1.51217
0.90	0	0	0.67631	1.36493	1.50851
0.95	0	0	0.64104	1.36268	1.50631
1.00	0	0	0	1.36193	1.50558

For the case of applied surface heat flux, the temperature variation at x=0 is significant in some practical applications. The computed temperature values, plotted in Figure 2, show that there is an initial jump at t=0 to be followed by other jumps after reflections. When $\tilde{\tau}_q = 4$, the wave speed is less than that when $\tilde{\tau}_q = 1$ and 1/4. Also, this figure indicates a second jump at $\tilde{\tau} = 4$, the wave front arrivals time from x=L surface. The solutions using Fourier heat conduction are also plotted in Figure 2. They show that the difference between these two solutions reduces as $\tilde{\tau}_q$ reduces; that is when the thermal wave speed increases. Before arrival of the front, when $t/\tau_q < 2/\sqrt{\tilde{\tau}_q}$, the temperature at x=0 is obtainable from the solution for semi-infinite bodies, that is,

$$\frac{kT(0,t)}{q_o\sqrt{\alpha t}} = \frac{e^{-\frac{\eta}{2}}}{\sqrt{\eta}}\left[(1+\eta)I_0\left(\frac{1}{2}\eta\right)+\tilde{t}I_1\left(\frac{1}{2}\eta\right)\right] \qquad \text{(X22B10-9)}$$

where $\eta = \tilde{t} / \tilde{\tau}_q$. The right side of this equation reduces to $2/\sqrt{\pi}$ as τ_q approaches a zero value for the Fourier heat conduction

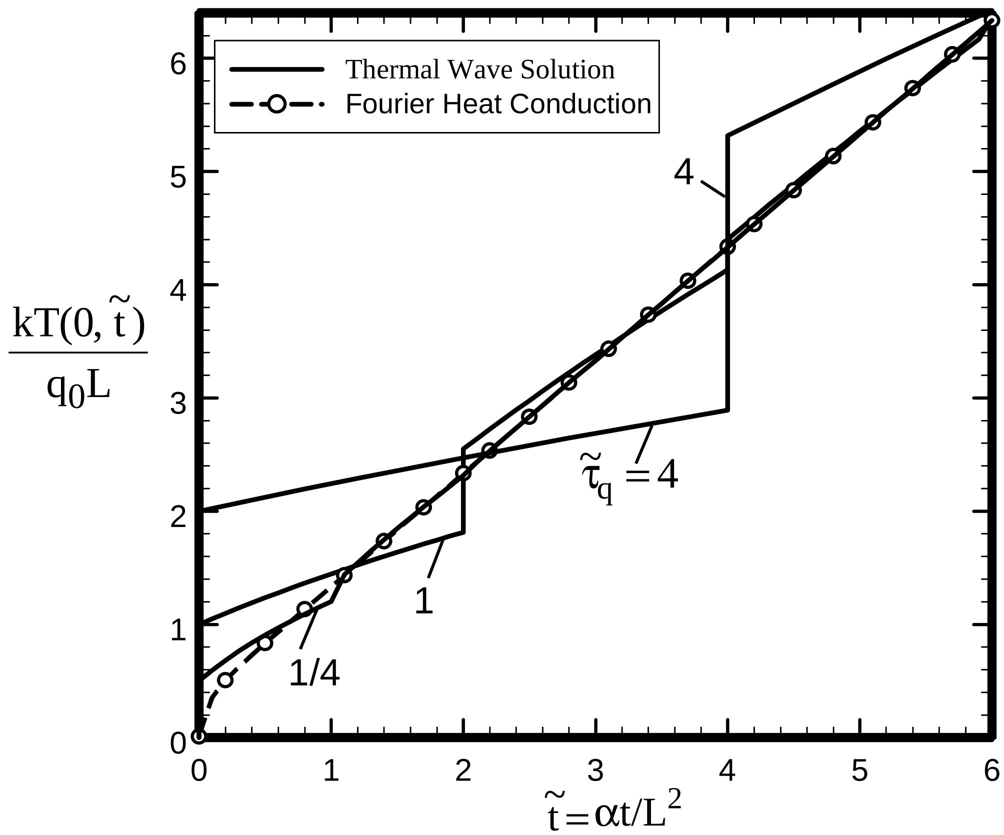

Figure 2. A comparison of the surface temperature solutions $\tilde{T}(0,\tilde{t})$ using thermal wave equation and Fourier heat conduction.

Reference: Haji-Sheikh, A., de Monte, Filippo, and Beck, James V., Temperature solutions in thin films using thermal wave Green's function solution equation, Int J. of Heat and Mass Transfer, Vol. 62 (2013) pp. 78-86.

Mathematica program for X22B10T00

In the following programs, "x" stands for $\tilde{x} = x / L$, "t" stands for $t / \tau_q = \tilde{t} / \tilde{\tau}_q$, and "tq" stands for $\tilde{\tau}_q$, "Fo" stands for $\tilde{\tau}$, and "Fomax" is the desired maximum value of $\tilde{\tau}$.

The inputs are:
the dimensionless location "x",
the value of the Fourier number "Fo",
the maximum value of the Fourier number "Fomax" (if needed),
the delay function $\tilde{\tau}_q$ represented by "tq" .

As a standard programming technique, it is convenient to form two external functions TX20B1T00 for temperature and qX20B1T00 for heat flux. A Mathematica program can performs symbolic and numerical determination of temperature and heat flux, as shown below. Next it is to specify the desired location "x" for *x/L*, the Fourier number "Fo", and "tq" to be inserted into the Mathematica program below that calls for the preceding external functions. As a test case, two sets of input data are selected with the same value of $\tilde{\tau}_q = 4$. The first set, produces temperature at the location *x/L*=1-10^{-5} when $\tilde{t} = 2$. Then, the process is repeated to get temperature and heat flux at *x/L*=1 when $\tilde{t} = 2 + 10^{-5}$. The obtained results show that there is a temperature jump and it changes by a factor of 2, after reflection; they are.

. $\tilde{x}$	$\tilde{\tau}$	$\tilde{T}$	$\tilde{q}$
0.99999 ,	**2. ,**	**1.55761,**	**0.778803**
1. ,	**2.00001 ,**	**3.11521,**	**0.**

```
(*External Functions*)
TX20B1T00[x_,t_]:=(tmp0=If[t>x,Exp[-t/2]*BesselI[0,Sqrt[t^2-
x^2]/2],0];intg=If[t>x,NIntegrate[Exp[-
tt/2]*BesselI[0,Sqrt[tt^2-x^2]/2],{tt,x,t}],0];Tmp=tmp0+intg;
  Return[Tmp])
qX20B1T00[x_,t_]:=(term=If[t>x,Exp[-
x/2],0];fnc=If[x>0&&t>x,NIntegrate[Exp[-
tt/2]*x*BesselI[1,Sqrt[tt^2-x^2]/2]/Sqrt[tt^2-
x^2]/2,{tt,x,t}],0];q=fnc+term;Return[q]);
```

```
(*The Basic Program*)
x=1-1/100000;Fo=2;tq=4;L=1;t=Fo/tq;
max=Round[t*Sqrt[tq]]+1;temp=0;q=0;sgn=-1;
Do[phi1=(x+2*m)/Sqrt[tq];phi2=(2*L+2*m*L-
x)/Sqrt[tq];temp=temp+(TX20B1T00[phi1,t]+TX20B1T00[phi2,t]);q=q+
(qX20B1T00[phi1,t]-
qX20B1T00[phi2,t]),{m,0,max}];temp=temp*Sqrt[tq];;
Print[N[x],",     ",N[t*tq],",    ",N[temp],",     ",N[Chop[q]]]
```

This program is extended below in order to compare computed values of temperature and heat flux from the thermal wave solution with those using Fourier-type heat conduction. The next computer program produced the data presented in Figure 2. Additional data are presented as the direct output of this program for $\tilde{\tau}_q = 1/25$. At this higher speed, the acquired temperature and heat flux values are the direct outputs of the Mathematica program below and they show moderate agreements with temperature and heat flux values using the Fourier heat conduction.

(*External Functions*)

```
TX20B1T00[x_,t_]:=(tmp0=If[t>x,Exp[-t/2]*BesselI[0,Sqrt[t^2-x^2]/2],0];intg=If[t>x,NIntegrate[Exp[-tt/2]*BesselI[0,Sqrt[tt^2-x^2]/2],{tt,x,t}],0];Tmp=tmp0+intg;
  Return[Tmp])
qX20B1T00[x_,t_]:=(term=If[t>x,Exp[-x/2],0];fnc=If[x>0&&t>x,NIntegrate[Exp[-tt/2]*x*BesselI[1,Sqrt[tt^2-x^2]/2]/Sqrt[tt^2-x^2]/2,{tt,x,t}],0];q=fnc+term;Return[q]);
TX22B10T0[x_,t_]:=(temp=0;NT=20;Do[bet=(n*Pi);coef=1/bet^2;temp=temp+2*coef*Cos[bet*x]*Exp[-t*bet^2],{n,1,NT}];Tquas=x^2/2-x+(t+1/3);temp=Tquas-temp;Return[temp])
qX22B10T0[x_,t_]:=(qchk=0;NT=20;Do[bet=(n*Pi);qchk=qchk-2*Sin[bet*x]*(Exp[-t*bet^2])/bet,{n,1,NT}];qchk=1-x-qchk;Return[qchk])
```

(*Input parameters to be used here*)

```
x=1/2;Fomax=1;tq=1/25;nmax=5;dt=(Fomax/tq)/nmax;dt0=x/Sqrt[tq];x=If[x 0,0/10^10,x];L=1;Print["Time","     Fo Num ","
TX22B1T00","     qX22B1T00","     TX22B10T0","
qX22B10T0"];ni=Round[1+1*dt0/(dt0+0.2)];dt0=dt0/ni;
Do[t=j*dt0;
 Print[N[t],",      ",N[t*tq],",
",N[TX20B1T00[x/Sqrt[tq],t]*Sqrt[tq]],",
",N[qX20B1T00[x/Sqrt[tq],t]],",
",N[TX22B10T0[x,t*tq]],",    ",N[qX22B10T0[x,t*tq],8]],{j,1,ni}]
Print[N[t],",     ",N[t*tq],",
",N[TX20B1T00[x/Sqrt[tq],t+1/10^8]*Sqrt[tq]],",
",N[qX20B1T00[x/Sqrt[tq],t+1/10^8]],",
",N[TX22B10T0[x,t*tq]],",
",N[qX22B10T0[x,t*tq]]];nmin=Round[t/dt];t=nmin*dt;
Do[t=t+dt;max=Round[t*Sqrt[tq]]+1;temp=0;q=0;Do[phi1=(x+2*m)/Sqrt[tq];phi2=(2*L+2*m*L-x)/Sqrt[tq];temp=temp+(TX20B1T00[phi1,t]+TX20B1T00[phi2,t]);q=q+(qX20B1T00[phi1,t]-qX20B1T00[phi2,t]),{m,0,max}];temp=temp*Sqrt[tq];
 Print[N[t],",       ",N[t*tq],",       ",N[temp],",
",N[Chop[q]],",       ",N[TX22B10T0[x,t*tq]],",
",N[qX22B10T0[x,t*tq]]],{j,1,nmax-nmin}]
```

Time	Fo Num	TX22B1T00	qX22B1T00	TX22B10T0	qX22B10T0
1.25,	0.05,	0.,	0.,	0.0153659,	0.88615580
2.5,	0.1,	0.,	0.,	0.0593109,	0.73724373
2.5,	0.1,	0.057301,	0.286505,	0.0593109,	0.737244
5.,	0.2,	0.162736,	0.443044,	0.158352,	0.588434
10.,	0.4,	0.358663,	0.507303,	0.358333,	0.512284
15.,	0.6,	0.558349,	0.49995,	0.558333,	0.501706
20.,	0.8,	0.758333,	0.499954,	0.758333,	0.500237
25.,	1.,	0.958333,	0.500003,	0.958333,	0.500033